Lecture XXXII: §8.5 (cont): radioactive decay. §9.1 Review of trigonometry

§1. Radioactive Decay:

A characteristic feature of radioactive materials is the decay in radioactive material. We write $x(t)$ = amount of material at time t

Rate of decay: $\frac{dx}{dt} = -kx$ with $k>0$ decay constant

Solution: $\boxed{x(t) = x_0 e^{-kx}}$ x_0 = amount of material at time 0.

Note: $x(t) \neq 0$ for all t if $x_0 \neq 0$, so radioactive materials NEVER completely decay.

Analog of double time is the half-time $t_{1/2}$ (time it takes for the substance to decay to half of its original amount)

$x(t_{1/2}) = \frac{1}{2} x_0$ gives $e^{-kt_{1/2}} = \frac{1}{2} = 2^{-1}$

$$-kt_{1/2} = \ln 2^{-1} = -\ln 2$$

$$\boxed{kt_{1/2} = \ln 2}$$

Remark: $x(t+t_{1/2}) = x_0 e^{-k(t+t_{1/2})} = \underbrace{x_0 e^{-kt}}_{x(t)} \underbrace{e^{-kt_{1/2}}}_{1/2} = \frac{x(t)}{2}$

So the initial time is irrelevant for computing $t_{1/2}$.

Example Cesium 137 decays to 20% in 10 years. What's its half-time?

Solution: $x_0 e^{-k10} = x(10) = \frac{20}{100} x_0 = \frac{x_0}{5}$ so $e^{-k10} = \frac{1}{5}$

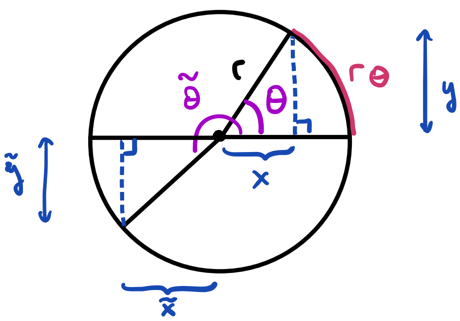
gives $-k10 = \ln \frac{1}{5} = -\ln 5 \rightsquigarrow k = \frac{\ln 5}{10}$

So $t_{1/2} = \frac{\ln 2}{k} = \frac{\ln 2}{\frac{\ln 5}{10}} = \boxed{\frac{10 \ln 2}{\ln 5}} \approx 4.3067$ years.

Main application: Radio carbon dating (Libby ~1940)

- Carbon 14 is in living things. It starts decaying right after death. It has a half-time of ~5600 years.
- **Example**: If a piece of old wood has half radioactivity from Carbon 14 as a living tree has, then it lived about 5600 years ago. If it has $\frac{1}{4}$ of radioactivity it lived 11,200 years ago, etc.

§2 Basics on Trigonometry:



- θ = measured in degrees between 0° & 360°
- θ = ———— radians ——— 0 & 2π

(1 radian = angle to describe an arc of the unit circle of length 1)

$$\begin{cases} \text{Area (sector)}: \pi r^2 \dfrac{\theta}{2\pi} = \pi r^2 \dfrac{\theta}{\pi} \\ \text{Length (arc)}: \theta r = 2\pi r \dfrac{\theta}{2\pi} \end{cases}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$ (polar coordinates) for $0 \leq \theta < 2\pi$ $\qquad r = \sqrt{x^2+y^2}$

We "redefine $\cos\theta$ & $\sin\theta$" via $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$ & $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$

θ = angle between the positive x axis and the segment joining $(0,0)$ & (x,y) (counterclockwise)

Example: $\pi < \tilde{\theta} < \frac{3}{2}\pi$ so $\tilde{x} = r\cos\tilde{\theta} < 0$ & $\tilde{y} = r\sin\tilde{\theta} < 0$.

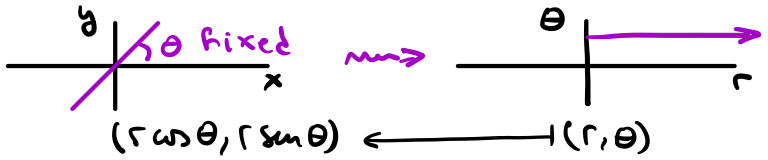
Periodicity: $\cos(\theta + 2\pi) = \cos(\theta)$ & $\sin(\theta + 2\pi) = \sin\theta$.

4 more trig functions $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot(\theta) = \dfrac{1}{\tan} = \dfrac{\cos\theta}{\sin\theta}$ (periodicity π)

$\sec\theta = \dfrac{1}{\cos\theta}$ $\qquad \csc(\theta) = \dfrac{1}{\sin\theta}$ (periodicity 2π)

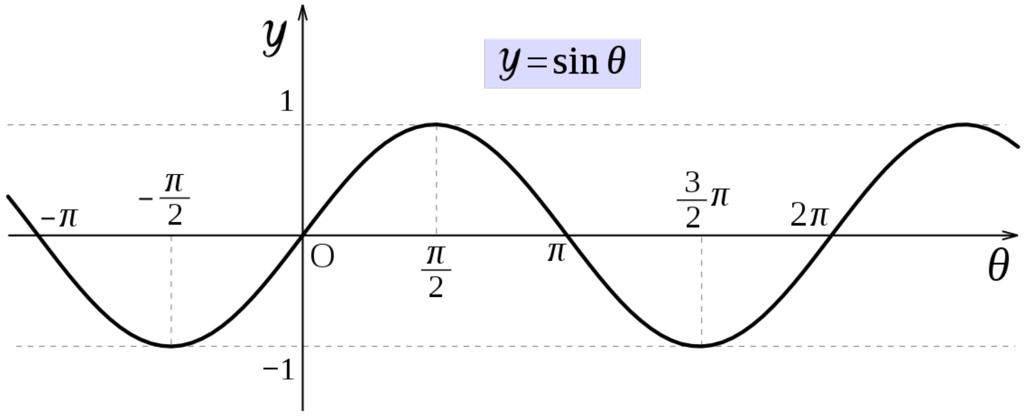
Parity
$$\left.\begin{aligned} \sin(-\theta) &= -\sin\theta \\ \tan(-\theta) &= -\tan\theta \end{aligned}\right\} \text{ODD}$$
$$\cos(-\theta) = \cos\theta \qquad \text{EVEN}$$

Note: $y = mx$ in cartesian coordinates becomes $\theta = $ constant in polar coordinates

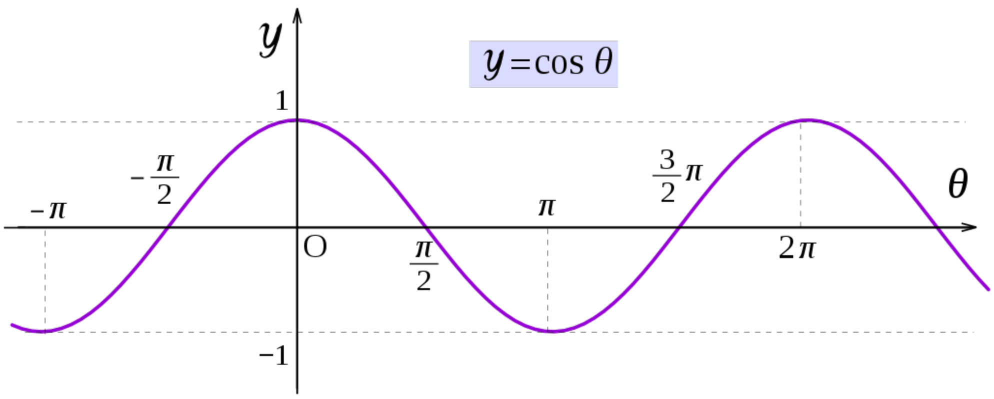


§3. Graphs: Source: Wikipedia

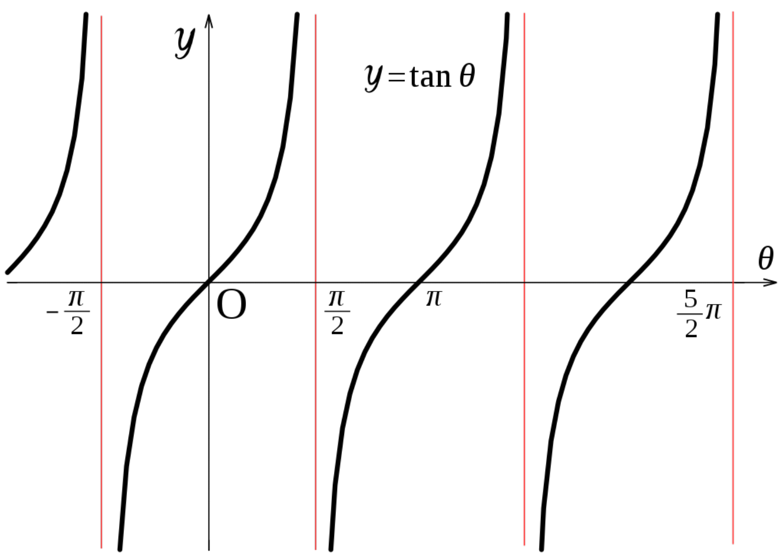
① $\boxed{\sin x}$



② $\boxed{\cos x}$



③ $\boxed{\tan x}$



§4. Trig Identities

① $\sin^2\theta + \cos^2\theta = 1$ $\quad (x^2 + y^2 = r^2 \Leftrightarrow r^2\cos^2\theta + r^2\sin^2\theta = r^2)$

$\tan^2\theta + 1 = \sec^2\theta \quad (= 1/\cos^2\theta)$

$1 + \cot^2\theta = \csc^2\theta \quad (= 1/\sin^2\theta)$

(2) Addition formulas:

$$(*)\begin{cases}\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi \\ \cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi\end{cases}$$

divide by $\cos\theta\cos\varphi$ both numerator & denominator

So $$\tan(\theta+\varphi) = \frac{\sin\theta\cos\varphi+\cos\theta\sin\varphi}{\cos\theta\cos\varphi-\sin\theta\sin\varphi} = \frac{\tan\theta+\tan\varphi}{1-\tan\theta\tan\varphi}$$

Why (*)? We need an aside on exponentials and complex numbers (Euler)

- Define i as $\sqrt{-1}$, meaning $i^2=-1$ ⇝ add $\sqrt{\ }$ of any negative real number!
- A complex number is an expression of the form $a+ib$ with a,b in $\mathbb{R}$

(Think of i as a placeholder)

Then $e^{a+ib} = e^a(e^{ib})$ ($z=a+ib$, a,b in $\mathbb{R}$ is a complex number)

$f(x)=e^{ix}$ satisfies $f'(x) = i e^{ix}$ (Chain Rule) so $f''(x) = i\,i\,e^{ix} = -e^{ix}$

Conclude: $f''(x) = -f(x)$

Now $\sin x$ & $\cos x$ also satisfy the same equation. They must be related (equation involves 2nd derivative & we have 3 solutions)

Now $f(x) = e^{ix}$ & $g(x) = \alpha\cos x + \beta\sin x$ solve the same differential equation. If $f(0)=g(0)$ & $f'(0)=g'(0)$ they must agree!

$f'(x) = ie^{ix}$ & $g'(x) = -\alpha\sin x + \beta\cos x$

We find α & β by convenient evaluation of e^{ix} & its derivative at $x=0$ (2 initial conditions)

$1 = e^0 = \alpha\cos 0 + \beta\cdot\sin 0 = \alpha$ so $\alpha=1$

$i = ie^0 = -\alpha\sin 0 + \beta\cos 0 = \beta$ so $\beta = i$

Conclude: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ (Euler's Theorem)

But $e^{i(\theta+\varphi)} \underset{\text{Laws of Exp}}{=} e^{i\theta}e^{i\varphi} = (\cos\theta+i\sin\theta)(\cos\varphi+i\sin\varphi)$

$= \cos\theta\cos\varphi + i\cos\theta\sin\varphi + i\sin\theta\cos\varphi + \underbrace{i^2}_{=-1}\sin\theta\sin\varphi$

regroup $= (\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi)$

Since $e^{i(\theta+\varphi)} = \cos(\theta+\varphi) + i\sin(\theta+\varphi)$ we recover the 2 addition formulas at once!