Lecture XXXIII:
§ 9.2 The Derivative of $\sin x$ & $\cos x$
§ 9.3 Integrals of $\sin x$ & $\cos x$
§ 9.4 The other 4 trigonometric functions
§ 9.5 The inverse trigonometric functions

Last Time: (1) We redefined $\cos\theta$ & $\sin\theta$ via polar coordinates & rediscovered the trig identity $\cos^2\theta + \sin^2\theta = 1$

(2) We used differential equations to check $e^{i\theta} = \cos\theta + i\sin\theta$
(2 functions with $f''_{(x)} = -f_{(x)}$ & same initial conditions $f_{(0)}=1$, $f'_{(0)}=i$)
& used this to prove the addition formulas.

$$(*)\begin{cases} \sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi \\ \cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi \end{cases}$$

Special cases: (useful for integration)

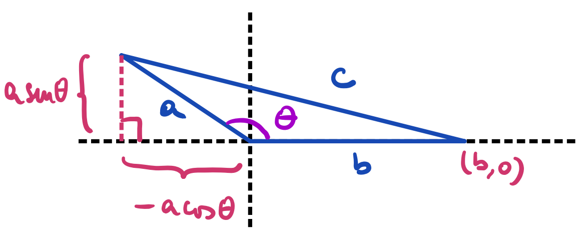
(1) Double angle:
$$\begin{cases} \sin 2\theta = 2\sin\theta\cos\theta \\ \cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta) \\ \qquad\qquad = 2\cos^2\theta - 1 \end{cases}$$

Also: $\cos 2\theta = (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$.

(2) Half-angle:
$$\begin{cases} 2\cos^2\theta = 1+\cos 2\theta \\ 2\sin^2\theta = 1-\cos 2\theta \end{cases}$$

gives $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$ & $\sin^2\theta = \frac{1-\cos\theta}{2}$

Law of cosines: Fix $\frac{\pi}{2} \leq \theta \leq \pi$



Then $\boxed{c^2 = a^2+b^2-2ab\cos\theta}$

Note: For $\theta = \pi/2$ This is usual Pythagoras's Theorem.

Q Why is the Law of cosines valid?

A Use trig identities!

$$c^2 = (a\sin\theta)^2 + (b - a\cos\theta)^2 = \underline{a^2\sin^2\theta} + b^2 + \underline{a^2\cos^2\theta} - 2ab\cos\theta$$
(add up to a^2)
$$= a^2+b^2-2ab\cos\theta$$

§2. Derivatives & Integrals:

$$\begin{cases} \frac{d}{d\theta}\sin\theta = \cos\theta & \frac{d}{d\theta}\tan\theta = \sec^2\theta & \frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta \\ \frac{d}{d\theta}\cos\theta = -\sin\theta & \frac{d}{d\theta}\cot(\theta) = \csc^2\theta & \frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta \end{cases}$$

(use the Quotient Rules for the last 4. For the first 2, use additive formulas + $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$)

$$\int \sin\theta\, d\theta = -\cos\theta + C \qquad \int \sec^2\theta\, d\theta = \tan\theta + C$$

$$\int \cos\theta\, d\theta = \sin\theta + C \qquad \int \csc^2\theta\, d\theta = -\cot\theta + C$$

$$\int \sec\theta\tan\theta\, d\theta = \sec\theta + C \qquad \int \csc\theta\cot\theta\, d\theta = -\csc\theta + C$$

Extra Formulas:

① $$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx \underset{u=\cos x}{=} \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

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② $$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx \underset{u=\sin x}{=} \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$$

③ $$\int \sec x\, dx = \ln|\sec(x) + \tan(x)| + C$$

Why? $$\int \sec x\, dx = \int \sec x\, \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{du}{u} = \ln|\sec(x) + \tan(x)| + C$$

$u = \sec x + \tan x$

$du = (\sec x\tan x + \sec^2 x)\, dx$

④ $$\int \csc(x)\, dx = -\ln|\csc(x) + \cot(x)| + C$$

Why? $$\int \csc(x)\, dx = \int \csc x\, \frac{\csc(x) + \cot(x)}{\csc(x) + \cot(x)}\, dx = \int \frac{\csc^2(x) + \csc(x)\cot(x)}{\csc(x) + \cot x}\, dx$$

$$= -\int \frac{du}{u} = -\ln|\csc(x) + \cot(x)| + C.$$

$u = \csc(x) + \cot(x)$

$du = (-\csc(x)\cot(x) - \csc^2(x))\, dx$

(5) $\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} \, dx = \frac{1}{2}x + \frac{\operatorname{sen} 2x}{4} + C$

$\int \operatorname{sen}^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx = \frac{x}{2} - \frac{\operatorname{sen} 2x}{4} + C$

Why? $\int \frac{\cos 2x}{2} \, dx = \int \frac{\cos u}{4} \, dx = \frac{\operatorname{sen} u}{4} + C = \frac{\operatorname{sen} 2x}{4} + C.$

$\hookrightarrow u = 2x$
$du = 2\,dx$

§1. Trigonometric substitution

GOAL: Find antiderivatives for the following 2 functions:

(1) $\int \frac{dx}{\sqrt{1-x^2}} = ??$ (2) $\int \frac{dx}{1+x^2} = ??$

We do this via Trigonometric substitution.

(1) Write $x = \operatorname{sen}(u)$ so $dx = \cos u \, du$

$\sqrt{1-x^2} = \sqrt{1-\operatorname{sen}^2 u} = \sqrt{\cos^2 u} = \cos u$

So $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos u \, du}{\cos u} = \int du = u + C$
$= (\operatorname{sen} x)^{-1} + C$

(2) Write $x = \tan(u)$ so $dx = \sec^2 u \, du$

$1+x^2 = 1+\tan^2 u = 1 + \frac{\operatorname{sen}^2 u}{\cos^2 u} = \frac{\cos^2 u + \operatorname{sen}^2 u}{\cos^2 u} = \frac{1}{\cos^2 u}$

So $\int \frac{dx}{1+x^2} = \int \frac{\sec^2 u \, du}{1/\cos^2 u} = \int \cos^2 u \frac{1}{\cos^2 u} \, du = \int du = u + C$
$= (\tan x)^{-1} + C$

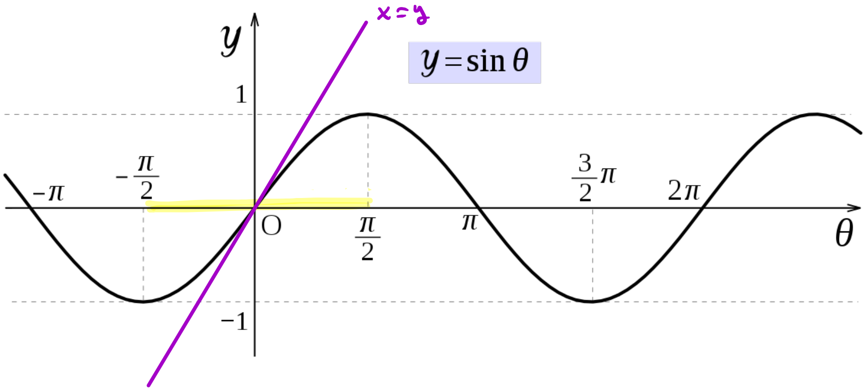
In both cases, we need to find ways to rewrite $x = \operatorname{sen} u$ or $x = \tan u$ as $u = u(x)$. This means finding inverses to $\operatorname{sen} x$ & $\tan x$.

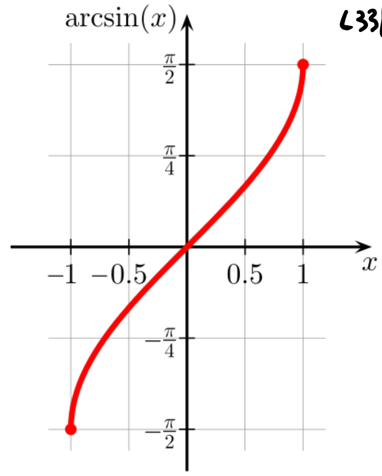
§2. Inverse sine: arcsin(x) or $\sin^{-1}(x)$

Recall $\sin(x)$ is periodic with period 2π

Image of sin $\times [-1, 1]$

Q How can we invert $\operatorname{sen} x$? A We MUST restrict its domain





restrict to $[-\frac{\pi}{2}, \frac{\pi}{2}]$
& reflect about $x=y$

Def: $\arcsin : [-1,1] \longrightarrow \mathbb{R}$ satisfies $\arcsin x = y$ if $\sin y = x$

Image $(\arcsin) = [-\frac{\pi}{2}, \frac{\pi}{2}]$

Domain $(\arcsin)$ = Image $\sin = [-1,1]$

By construction, the graph has tangent lines everywhere, so arcsin is differentiable in $(-1,1)$. We can compute $\frac{d}{dx}\arcsin x$ by implicit differentiation:

Proposition: $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$

Q: Why? $y = \arcsin x$ satisfies $\sin y = x$

So $\frac{d}{dx}$ gives: $(\cos y)\, y' = 1 \rightsquigarrow y' = \frac{1}{\cos y}$ ($\cos y \neq 0$ if $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$)

But $\cos y > 0$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ so $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$ so

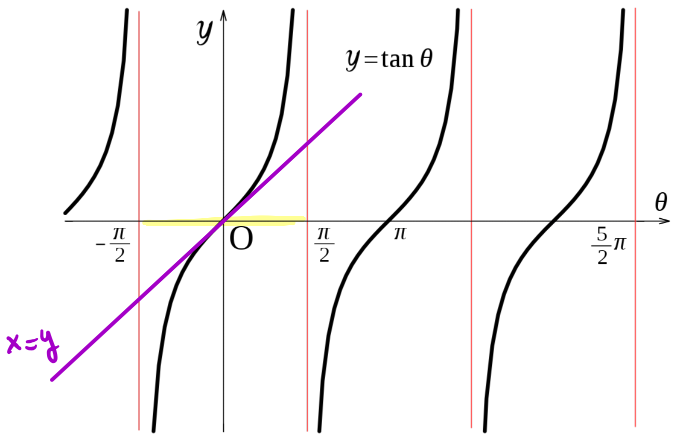
$y' = \frac{1}{\sqrt{1-x^2}}$ as we predicted.

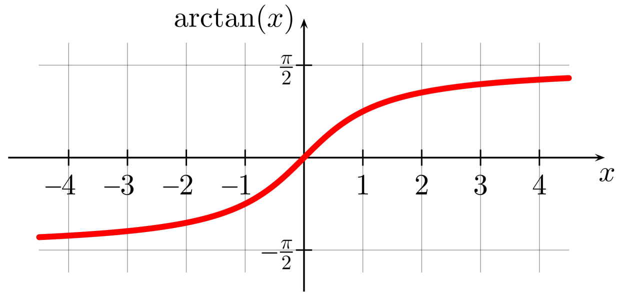
Consequence: $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$

§3. Inverse Tangent: $\arctan x$ or $\tan^{-1}(x)$.

Recall
- $\tan x$ is periodic with period π
- vertical asymptotes at $\pi/2, 3\pi/2, \ldots$ $-\pi/2, -3\pi/2, \ldots$
- Image $= \mathbb{R}$

$\rightsquigarrow$ Need to restrict the domain of $\tan x$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$





restrict to $(-\frac{\pi}{2}, \frac{\pi}{2})$
& reflect about $x=y$

Definition $\arctan : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ where $\arctan x = y$ if $\tan y = x$

Again, the graph has tangent lines everywhere, so arctan is differentiable.

Proposition: $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

Q: Why? Again, we use implicit differentiation on $\tan y = x$

$$(\sec^2 y) \cdot y' = 1$$

$$\frac{y'}{\cos^2 y} = 1 \quad \text{and} \quad \frac{1}{\cos^2 y} = \frac{\sin^2 y + \cos^2 y}{\cos^2 y} = \tan^2 y + 1 = x^2+1$$

So $y' = \frac{1}{1/\cos^2 y} = \frac{1}{1+x^2}$.

Consequence: $\int \frac{1}{1+x^2}\, dx = \arctan x + C$

Examples: ① $\frac{d}{dx}\left(\arcsin\left(\frac{x}{5}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}} \cdot \frac{1}{5} = \frac{1}{\sqrt{25-x^2}}$

② $\frac{d}{dx}\left(\arctan\left(\sqrt{1+x^2}\right)\right) = \frac{1}{1+(1+x^2)} \cdot \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{(2+x^2)\sqrt{1+x^2}}$

③ $\int \frac{dx}{1+16x^2} \underset{u=4x}{=} \int \frac{1}{1+u^2} \frac{du}{4} = \frac{1}{4}\arctan u + C = \frac{1}{4}\arctan(4x) + C$

④ $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{u}{\sqrt{\frac{1}{u^2}-1}} \frac{(-du)}{u^2} = \int \frac{-du}{\frac{u}{u}\sqrt{1-u^2}} = -\int \frac{du}{\sqrt{1-u^2}}$

$u = \frac{1}{x}$, $x^2 = \frac{1}{u^2}$

$du = \frac{-dx}{x^2}$, so $dx = -\frac{du}{u^2}$

$= \arcsin u + C$
$= \arcsin(1/x) + C$.