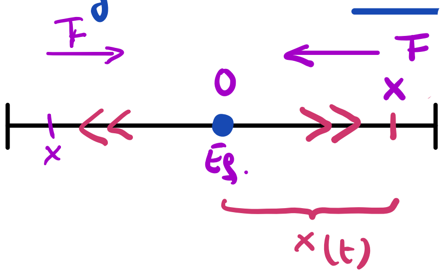
Lecture XXXIV: §9.6 Simple Harmonic Motion.

TODAY: We'll solve differential equations modeling:

1. the motions of vibrations producing sound
2. oscillations or waves of periodic motions.

§1 Simple Harmonic Motion:

Definition: If an object or a particle moves back & forth in a straight line (say the x-axis), so that the force required to move it back to equilibrium ($x=0$) is proportional to the distance away from equilibrium, we say we have a simple harmonic motion (SHM)



$$F(t) = m\,a(t) = m\frac{d^2x}{dt^2} = -k\,x(t) \quad \text{for } k>0$$

(restorative force)   (if $k=0$, no movement)

gives $\quad \dfrac{d^2x}{dt^2} = -\dfrac{k}{m}\,x(t)$

Now: $k, m>0$ so $c = \frac{k}{m} > 0$ we write $\boxed{c = a^2 > 0}$ for some a, to ensure it is positive!

Conclusion: The equation governing the SHM is

$$(*) \quad \boxed{\frac{d^2x}{dt^2} + a^2 x(t) = 0 \quad \text{for } a>0} \quad \text{(degree 2)}$$

Initial conditions: $\begin{cases} x(t_0) = x_0 & \text{(initial position)} \\ x'(t_0) = v_0 & \text{(initial velocity, typically } v_0=0) \end{cases}$

EXAMPLE: $a=1$ gives $x'' + x = 0 \rightsquigarrow$ 2 building blocks for solutions namely: $\sin t$ & $\cos t$. **General solution:** $x = \alpha \sin t + \beta \cos t$
(α, β parameters)

Theorem: The solutions to the SHM equation (*) are of the form:

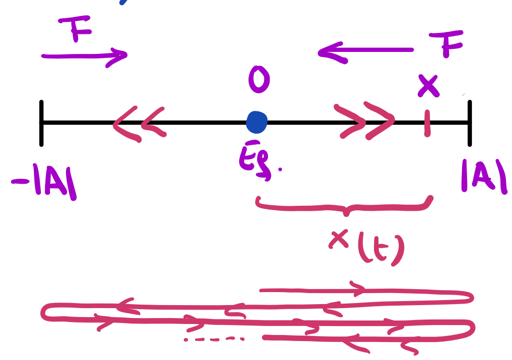
$$x(t) = A \sin(at+b) \quad \text{for some } a, b \text{ in } \mathbb{R}$$

Names:
- $|A|$ = amplitude
- $T = \frac{2\pi}{a}$ = period (= smallest time it takes to return to a given position.

Indeed, $x(t+T) = A \sin(a(t+T)+b) = A \sin(at+b+2\pi)$
$= A \sin(at+b) = x(t)$

- $f = \frac{1}{T} = \frac{a}{2\pi}$ = frequency (= number of cycles per second)

Note: By construction, the movement oscillates between $-|A|$ & $|A|$



It takes $\frac{2\pi}{a}$ seconds to go over the cycle once.

Why? We will give 2 arguments:

(1) Write $\frac{d^2x}{dt^2} = a(t) = \frac{dv}{dt}$ & use Chain Rule on $v = v(x)$

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$ (as we did for Escape Velocity calculations)

(SHM) equations $\frac{d^2x}{dt^2} + a^2 x = 0$ becomes:

$$\frac{dv}{dx} v + a^2 x = 0$$

We can solve this via separation of variables:

$$dv \cdot v = -a^2 x \, dx$$

$$\int v \, dv = \int -a^2 x \, dx$$

$$\frac{v^2}{2} = \frac{-a^2 x^2}{2} + C$$

Conclude: $v^2 + a^2 x^2 = 2C \geq 0$ is constant

Next, we use the initial conditions to determine C & so v:

At $t = t_0$ we get $\quad v_0^2 + a^2 x_0^2 = 2C$

So $\quad v^2(x) = v_0^2 + a^2 x_0^2 - a^2 x^2 = v_0^2 + a^2(x_0^2 - x)$

$$= a^2 \Big(\underbrace{\left(\frac{v_0}{a}\right)^2 + x_0^2}_{=:\, A^2} - x^2 \Big)$$

$\rightsquigarrow$ x cannot exceed $|A|$ $\quad (v^2 = a^2(A^2 - x^2) \geq 0)$

Then $\quad \dfrac{dx}{dt} = v = \pm a \sqrt{A^2 - x^2}$ $\quad$ (sign depends on the direction!)

Once again, we can use separation of variables to solve for x:

$$\frac{dx}{\sqrt{A^2 - x^2}} = \pm a \, dt$$

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \int \pm a \, dt = \pm a t + b$$

(LHS) can be computed by substitution $u = \frac{x}{A}$

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \int \frac{dx}{\sqrt{A^2\left(1 - \left(\frac{x}{A}\right)^2\right)}} = \int \frac{du}{\sqrt{1 - u^2}} = \arcsin(u) = \arcsin\left(\frac{x}{A}\right)$$

$u = \frac{x}{A}$

$du = \frac{dx}{A}$

Conclude: $\quad \arcsin\left(\frac{x}{A}\right) = \pm a t + b \quad \rightsquigarrow \quad x = A \sin(\pm a t + b)$

- If $v > 0$, then $x = A \sin(at + b)$
- If $v < 0$, then $x = A \sin(-at + b) = (-A) \sin(at - b)$

$\hookrightarrow \sin(-w) = -\sin(w)$

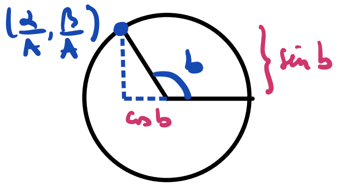
So the sign of A depends on the direction of the movement, but the formula for $x(t)$ looks the same in both situations.

② For our 2nd approach we need the following claim:

CLAIM: Any solution to $(*)$ has the form $\quad x(t) = \alpha \sin(at) + \beta \cos(at)$

Since we don't want the trivial solution $x(t) \equiv 0$, we can assume either α or $\beta \neq 0$.

We set $\boxed{A} = \sqrt{\alpha^2+\beta^2} > 0$, so the point $\left(\frac{\alpha}{A}, \frac{\beta}{A}\right)$ lies on the unit circle



Using polar coordinates, we can find b with $\frac{\alpha}{A} = \cos b$ & $\frac{\beta}{A} = \sin b$

Conclude:
$$\begin{aligned} x(t) &= \alpha \sin(at) + \beta \cos(at) \\ &= A\left(\frac{\alpha}{A}\sin(at) + \frac{\beta}{A}\cos(at)\right) \\ &= A\left(\cos b \sin(at) + \sin b \cos(at)\right) \\ &= A \sin(at+b) \end{aligned}$$

Note: The only thing we need is $A \neq 0$. This expression will also be a solution to (SHM) even if $A<0$.

Key: $A = \sqrt{\alpha^2+\beta^2}$ if $x(t) = \alpha\sin(at) + \beta\cos(at)$ is a solution to (SHM) $\rightsquigarrow$ $T = \frac{2\pi}{a}$ & $f = \frac{1}{T} = \frac{a}{2\pi}$

§ 2. An alternative solution:

Proposition: Any solution to $y'' + a^2 y = 0$ for $a \neq 0$ has the form
$$y(t) = \alpha \sin(at) + \beta \cos(at)$$

Q: Why? We argue in 4 steps

STEP 1: If $f(t)$, $g(t)$ are solutions & α, β are real numbers,
$$y(t) = \alpha f(t) + \beta g(t)$$
is also a solution. (Note $\sin(at)$ & $\cos(at)$ are solutions)

This is easy to check:
$$y' = \alpha f' + \beta g' \rightsquigarrow y'' = \alpha f'' + \beta g''$$
$$+ \quad a^2 y = a^2\alpha f + a^2 \beta g$$
$$y'' + a^2 y = \alpha(\underbrace{f'' + a^2 f}_{=0}) + \beta(\underbrace{g''+a^2 g}_{=0}) = 0$$

STEP 2: If $y = f(t)$ is a solution, then $a^2 f_{(t)}^2 + (f'_{(t)})^2$ is a constant.

Why? Check that $h(t) = a^2 f_{(t)}^2 + (f'_{(t)})^2$ has derivative 0.

$$\frac{d}{dt}\left(a^2 f_{(t)}^2 + (f'_{(t)})^2\right) = a^2 2 f(t) f'(t) + 2 f' \cdot f'' = 2f'(\underbrace{a^2 f + f''}_{=0}) = 0.$$

STEP 3: If $y(t)$ is a solution & $y(0) = y'(0) = 0$, then $y_{(t)} = 0$ for all t.

Why? We use STEP 2: $a^2 y(t)^2 + (y'(t))^2 = \text{constant}$ & we get the constant from setting $t=0$. We get $a^2 \cdot 0^2 + 0^2 = 0$.

So $\underbrace{a^2}_{\geq 0} \underbrace{y^2}_{\geq 0} + \underbrace{(y')^2}_{\geq 0} = 0$. This forces $a y_{(t)} = 0$ & $y'(t) = 0$

Since $a \neq 0$, we get $y(t) = 0$ for all t

STEP 4: Pick a solution $y(t)$ & write

$$f(t) = y(t) \underbrace{- \underbrace{\underbrace{\tfrac{1}{a} y'(0)}_{=\alpha} \sin(at) - \underbrace{y(0)}_{=\beta} \cos(at)}_{\text{solution by STEP 1}}}_{\text{solution by STEP 1}}$$

Now $f'(t) = y'(t) - \frac{1}{a} y'(0) \cos(at)\, a - y(0)(-\sin(at) \cdot a)$

$= y'(t) - y'(0) \cos(at) + a\, y(0) \sin(at)$

In particular $f(0) = y(0) - \frac{1}{a} y'(0) \sin(0) - y(0) \cos(0)$

$= y(0) - 0 - y(0) \cdot 1 = 0$

$f'(0) = y'(0) - y'(0) \cos(0) + a\, y(0) \sin(0) = 0.$

By STEP 3, we get $f(t) = 0$ (solution with trivial initial conditions)

Conclude: $y(t) = \underbrace{\boxed{\frac{1}{a} y'(0)}}_{=\alpha} \sin(at) + \underbrace{\boxed{y(0)}}_{=\beta} \cos(at)$

Exercise Find the amplitude & frequency of the SHM of a particle with trajectory $x(t) = 3\sin 2t + 4\cos 2t$. Determine its maximal velocity.

Solution: Recall: we need to write $x(t) = A\sin(at+b)$

Differential equation satisfied by $x(t)$ is $x''(t) + a^2 x(t) = 0$

Check the given trajectory solves this:

$$x' = 6\cos 2t + 8(-\sin 2t)$$

$$x'' = -12\sin 2t + 16(-\cos 2t) = \underbrace{-4(3\sin 2t + 4\cos 2t)}_{=-4x}$$

So $a^2 = 4$. Take $\boxed{a=2}$

$\rightsquigarrow$ Period: $T = \frac{2\pi}{a} = \boxed{\pi}$ & frequency: $f = \frac{1}{T} = \boxed{\frac{1}{\pi}}$

To find A & b we need to pick sample points:

$t=0$ gives: $x(0) = 3\cdot 0 + 4\cdot 1 = 4 = A\sin b$

$t = \frac{\pi}{4}$ —— $x(\frac{\pi}{4}) = 3\cdot 1 + 4\cdot 0 = 3$

$$= A\sin(2\tfrac{\pi}{4} + b) = A\sin(\tfrac{\pi}{2} + b) = A\cos b$$

So $\left.\begin{array}{l} A\sin b = 4 \\ A\cos b = 3 \end{array}\right\} \rightsquigarrow A^2 = A^2(\sin^2 b + \cos^2 b) = 4^2 + 3^2 = 25$

So $A = \pm 5$

$\rightsquigarrow$ Amplitude: $|A| = 5$. (Approach ②: $A = \sqrt{\alpha^2+\beta^2} \rightsquigarrow 5$)

- Velocity: $x' = 6\cos 2t + 8(-\sin 2t)$

Also $x' = 2A\cos(2t+b)$ $\rightsquigarrow$ maximal $x' = |2A| = 10$

In general: Maximal velocity for SHM $x(t) = A\sin(at+b)$ is $a|A|$ since $x'(t) = aA\cos(at+b)$