Lecture XXXV : §9.6 Simple Harmonic Motion II. The pendulum

Recall: Differential equation governing simple harmonic motion is

(SHM) $$\boxed{x''(t) + a^2\, x(t) = 0 \qquad \text{for } a>0}$$

Initial conditions: $x(t_0) = x_0$ (initial position)
$x'(t_0) = v_0$ (initial velocity, typically 0)

Two ways of writing down solutions:

METHOD 1: $x(t) = A \sin(at+b)$ where A, b are parameters

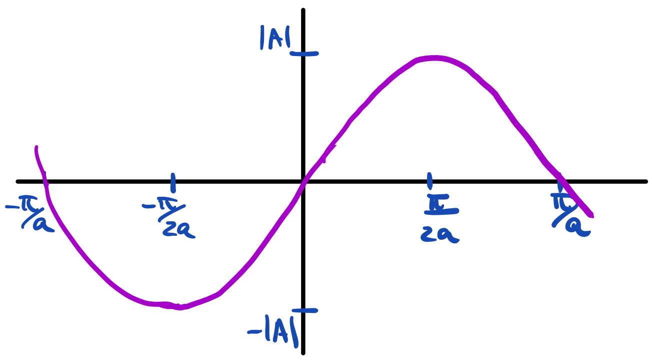
Names: $|A|$ = amplitude, $T = \frac{2\pi}{a}$ = period = smallest time it takes to return to a position

$f = \frac{1}{T}$ = frequency

METHOD 2: $x(t) = \alpha \sin(at) + \beta \cos(at)$ where α, β are parameters

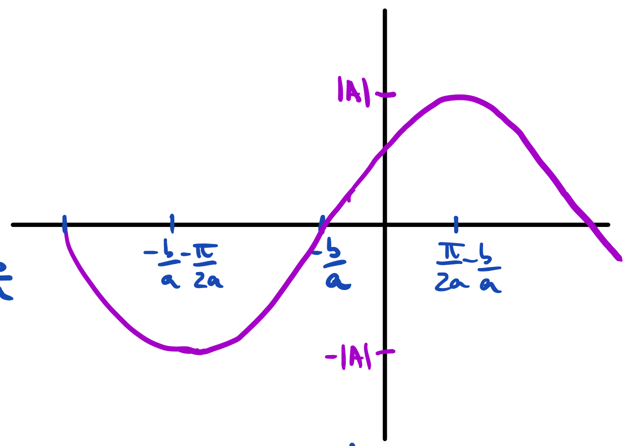
Connection: $A = \sqrt{\alpha^2+\beta^2}$

Q: Graph of $x(t)$?



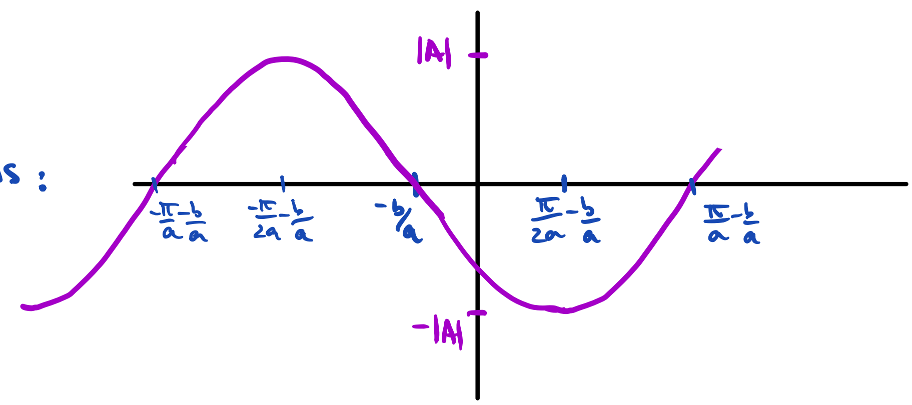
$b=0$ & $A>0$

shift by $-\frac{b}{a}$



$A>0$, any b

If $A<0$ rotate about x-axis:

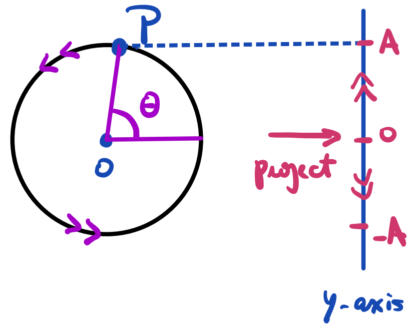


If $A=0$, we get $x(t)=0$ (boring solution!)

Remark: $x(t) = A\sin(at+b) = A\cos\left(at+b-\frac{\pi}{2}\right)$ (because $\sin(u) = \cos\left(u-\frac{\pi}{2}\right)$)

§1 Geometric perspective:

Assume $A>0$ & that a particle p moves around a circle of radius A with constant angular velocity : $\frac{d\theta}{dt} = a \ \frac{rad}{sec}$ (a = constant, $a>0$)



Then $\theta = at + b$ by integration.

So in polar coordinates P corresponds to (A, θ)

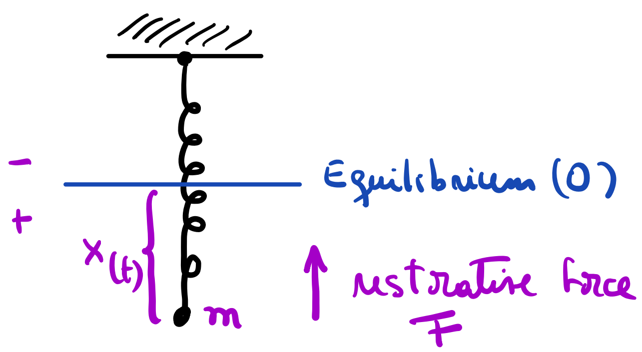
In cartesian coordinates $P = (x, y)$ with

$$x = A\cos\theta = A\cos(at+b)$$
$$y = A\sin\theta = \boxed{A\sin(at+b)}$$

So : the projection of the trajectory to the y-axis describes (SHM)

§2. Examples:

① SHM for a Spring:



Assume no air resistance (negligible)

Put an object of mass m at the end of a spring and let it stretch until it reaches equilibrium

Pull the spring away from the ceiling & let go of it (so $v_0 = 0$)

Newton's Law : $F = m\,a(t) = m\frac{d^2x}{dt^2} = -kx$ where

k is spring constant ($k>0$)

So $\frac{d^2x}{dt^2} = -\frac{k}{m}x = -a^2 x$ with $a = \sqrt{\frac{k}{m}}$

Initial conditions $\begin{cases} x(0) = x_0 \\ \frac{dx}{dt}(0) = v_0 = 0 \end{cases}$

We know $x(t) = \alpha\sin(at) + \beta\cos(at)$

So $x'(t) = a\alpha\cos(at) - \beta a\sin(at)$ gives

$x'(0) = 0 = a\alpha\cos(0) - 0 = a\alpha$ so $\alpha = 0$ $(a>0)$

Then $x(t) = \beta\cos(at)$

Now $x(0) = x_0$ gives $\beta = x_0$.

Conclude: $x(t) = x_0\cos\left(\sqrt{\frac{k}{m}}\,t\right) = x_0\sin\left(\sqrt{\frac{k}{m}}\,t + \frac{\pi}{2}\right)$

Amplitude $= |A| = x_0 > 0$ (initial position)

Period: $T = \frac{2\pi}{a} = 2\pi\sqrt{\frac{m}{k}}$

Frequency: $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Observations: • If the stiffness k of the spring increases, then the frequency increases

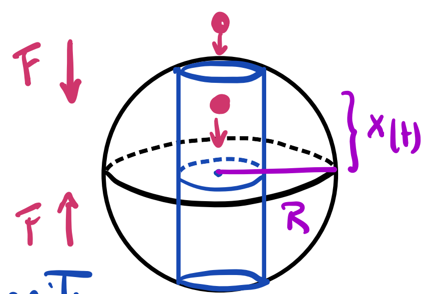
• If the mass m increases, then the frequency decreases.

These observations agree with the expectations we have based on experimental evidence.

② Hole in the Earth:

• We view the Earth as a sphere of radius $R = 4000$ mi.

• We bore a tunnel straight through the center of the Earth from north to south poles



• We drop a body of mass m into the tunnel.

GOAL: Find the motion of the body induced by gravity

• The force of gravity acts as if all the Earth was concentrated at its center

Model : $F_{(t)} = m\, x''_{(t)} = -k\, x$ gives $x'' + \frac{k}{m} x = 0$

At the surface $F = -mg = -kR$ $(x = R)$ so $\frac{k}{m} = \frac{g}{R}$

⇝ Equation is $\boxed{x'' + \frac{g}{R} x = 0}$ (independent of the mass!)

Conclusion : The object will exhibit SHM (if we disregard air friction etc)

- $a = \sqrt{\frac{g}{R}}$
- Period : $\frac{2\pi}{a} = 2\pi\sqrt{\frac{R}{g}} \approx 89$ minutes $\left(\begin{array}{l} R \sim 4000 \text{ mi} \\ g \sim 32 \text{ ft/s}^2 \\ 1\text{mi} = 5280 \text{ ft} \end{array}\right)$
- Time to get to the center : $\frac{89}{4} = 22$ minutes

Q Amplitude ?

A : Amplitude = radius of the Earth

Why? $x(t) = A \sin(at+b)$ & $x'(t) = aA\cos(at+b)$

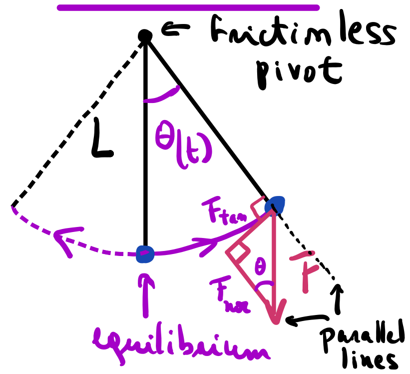
$x(0) = A \sin b = R$ $x'(0) = aA \cos b$

But $x'(0) = 0$ forces $\cos b = 0$ so $\sin b = \pm 1$.

So $R = |R| = |A \sin b| = |A|$ (as in the spring example!)

⚠ This will not be true if $x'(0) \neq 0$.

③ Pendulum :



Setup : a bob (weight) suspended at the end of a light string of length L

We allow the bob to switch back & forth under the action of gravity.

$S(t) = L\,\theta(t)$

$F = mg$ is not in the direction of the movement. For this we need to split

F into 2 components ① F_{tan} = component tangential to the movement

② F_{norm} = ——— perpendicular ———

$$F_{tan} = -F \sin\theta = -mg \sin\theta$$

⇝ Equation becomes $\dfrac{d^2 s}{dt^2} = \dfrac{d^2}{dt^2}\left(L\,\theta(t)\right) = L\dfrac{d^2\theta}{dt^2}$

Newton's Law : $\underset{\overset{\parallel}{-mg\sin\theta}}{F_{tan}} = m\dfrac{d^2 s}{dt^2} = mL\,\theta''$

Conclusion : $$\theta'' + \frac{g}{L}\sin\theta = 0$$

This is not the differential equation for (SHM) but it is very close to it when θ is closed to 0 since $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$

Near 0, the equation is close to $\theta'' + \frac{g}{L}\theta = 0$, so we get $\theta = A\sin\left(\sqrt{\frac{g}{L}}\,t + b\right)$ with period $T = 2\pi\sqrt{\frac{L}{g}}$.

⚠ This is ONLY an approximation! In reality, the period depends on the amplitude. This explains the "circular error" in pendulum clocks. (In a freely swinging pendulum, following a circular path under the influence of gravity take longer to traverse a large arc than a small arc)