Lecture XXXVI: §10.1 The basic formulas
§10.2 The method of substitution
§10.3 Certain trigonometric integrals

§1. The Basic Formulas:

Definition: An elementary function is one built from $x^a, e^x, \ln(x)$, $\sin(x)$, $\cos(x)$, $\arcsin(x)$, $\arctan(x)$ & constants

built from = add, multiply, divide, substract & compose.

Example: $\arctan\left(\dfrac{\ln(x^2+\cos^2(x))}{e^x+\sin\sqrt{x^{2.5}+1}}\right)+1$

Note: ① Simple rules for differentiation of building blocks + Chain / Product / Quotient Rules give easy ways to determine the derivatives of elementary functions. Moreover, these derivatives are once again elementary functions.

② Integration is more subtle: there is NO systematic way to do it & the output need not be an elementary function!

$\rightsquigarrow$ We have a recognition problem: (1) what method should we use?
(2) how to apply it?

Example (Appendix A9)

$Li(x) = \int_2^x \frac{dt}{\ln t}$ & $\int_0^x e^{-t^2}\,dt$ are not elementary functions

③ 15 basic formulas to use (pages 335 & 336 of textbook & handout)

§2. Method of substitution:

Substitution is the analog of the Chain Rule for integration $\rightsquigarrow$ recognition problem!

Substitution Rule: $\displaystyle\int_a^b f'(g(t))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du \underset{(FTC)}{=} f(u)\Big|_{g(a)}^{g(b)}$

$u = g(t)$ $\quad$ $t=a \rightsquigarrow u=g(a)$
$du = g'(t)\,dt$ $\quad$ $t=b \rightsquigarrow u=g(b)$

$= f(g(b)) - f(g(a))$

EXAMPLES ① $\displaystyle\int_0^2 x e^{-x^2}\,dx \underset{\substack{u=-x^2\\ du=-2x\,dx}}{=} \int_0^{-4} e^u \frac{du}{-2} = \frac{1}{2}\int_{-4}^{0} e^u\,du = \frac{1}{2}\left(e^u\Big|_{-4}^{0}\right) = \frac{1}{2}\left(1-e^{-4}\right)$

$x=0 \leadsto u=0$
$x=2 \leadsto u=-4$

② $\displaystyle\int_0^{\pi/2} \frac{\cos x\,dx}{\sqrt{1+\sin x}} \underset{\substack{u=1+\sin x\\ du=\cos x\,dx}}{=} \int_1^2 \frac{du}{\sqrt{u}} = 2u^{1/2}\Big|_1^2 = 2\cdot(\sqrt{2}-1).$

$x=0 \leadsto u=1$
$x=\frac{\pi}{2} \leadsto u=2$

Note: substituting $\cos x$ or $\sqrt{1+\sin x}$ won't work!

③ $\displaystyle\int_e^{e^2} \frac{dx}{x\ln x} \underset{\substack{u=\ln x\\ du=\frac{dx}{x}}}{=} \int_1^2 \frac{du}{u} = \ln u\Big|_1^2 = \ln 2 - \ln 1 = \ln 2.$

$x=e \leadsto u=1$
$x=e^2 \leadsto u=2$

④ $\displaystyle\int_0^{\sqrt{2}} \frac{dx}{\sqrt{9-4x^2}} = \int_0^{\sqrt{2}} \frac{dx}{3\sqrt{1-\left(\frac{2x}{3}\right)^2}} \underset{\substack{u=\frac{2}{3}x\\ du=\frac{2}{3}dx}}{=} \frac{1}{3}\int_0^{\frac{2}{3}\sqrt{2}} \frac{dx}{\sqrt{1-u^2}} = \frac{1}{3}\arcsin(u)\Big|_0^{\frac{2}{3}\sqrt{2}}$

$x=0 \leadsto u=0$
$x=\sqrt{2} \leadsto u=\frac{2}{3}\sqrt{2}<1$

$\displaystyle = \arcsin\left(\frac{2}{3}\sqrt{2}\right) - \arcsin 0 = \arcsin\left(\frac{2\sqrt{2}}{3}\right).$

⑤ $\displaystyle\int_0^{\sqrt{2}} \frac{x\,dx}{\sqrt{9-4x^2}} = \frac{1}{3}\int_0^{\sqrt{2}} \frac{x\,dx}{3\sqrt{1-\left(\frac{2x}{3}\right)^2}} \underset{\substack{u=1-\frac{4x^2}{9}\\ du=-\frac{8}{9}x\,dx}}{=} \int_1^{1/9} \frac{-9}{3\cdot 8}\frac{du}{u^{1/2}} = \frac{3}{8}\int_{1/9}^{1}\frac{du}{u^{1/2}} = \frac{3}{8}\,2u^{1/2}\Big|_{1/9}^{1}$

$x=0 \leadsto u=1$
$x=1 \leadsto u=\frac{1}{9}$

$\displaystyle = \frac{3}{4}\left(1-\frac{1}{3}\right) = \frac{1}{2}$

§3. Certain trigonometric examples:

GOAL: Find a method for integrating these 3 functions:

① $\int \sin^m(x)\cos^n(x)\,dx$

② $\int \tan^m(x)\sec^n(x)\,dx$ for $m,n\geq 0$ integers.

③ $\int \cot^m(x)\csc^n(x)\,dx$

Why? $\dfrac{d\sin x}{dx} = \cos x$ $\qquad \dfrac{d\tan x}{dx} = \sec^2(x)$ $\qquad \dfrac{d\cot(x)}{dx} = -\csc^2(x)$

$\leadsto$ Answers to ①, ②, ③ will depend on the parity of m & n

EXAMPLES: ① $(n=1)$ $\displaystyle\int \sin^m(x)\cos x\,dx \underset{u=\sin x}{=} \int u^m\,du = \frac{(\sin x)^{m+1}}{m+1} + C$

② $(n=2)$ $\int \tan^m(x)\,\sec^2(x)\,dx \underset{\substack{\downarrow \\ u=\tan x \\ du=\sec^2 x\,dx}}{=} \int u^m\,du = \frac{\tan^{m+1}(x)}{m+1} + C$

③ $(n=2)$ $\int \cot^m(x)\,\csc^2(x)\,dx \underset{\substack{\downarrow \\ u=\cot x \\ du=-\csc^2(x)dx}}{=} -\int u^m\,du = -\frac{\cot^{m+1}(x)}{m+1} + C$

Q: What about other values of n?

A We'll show the answer for ①. The other 2 are analogous. (next lecture)

We analyze 2 cases: even / odd combinations:

$\boxed{\text{CASE A: either } m \text{ or } n \text{ are ODD (eg } 1,3,5,7\ldots)}$

Trick: Use trig identities to turn integrand into sums of functions of the form

(1) $\cos^\ell(x)\sin(x)$ (if m is ODD) or (2) $\sin^\ell(x)\cos(x)$ (if n is ODD)

Why is this good? These terms can be integrated by a simple substitution!

How? We show it for (2) & then for (1):

- If n is ODD, we can write n as $n=2k+1$ for some integer $k\geq 0$.

Then $\cos^n x = \cos^{2k+1}(x) = \cos(x)\cos^{2k}(x) = \cos(x)\left(\cos^2(x)\right)^k = \cos(x)\left(1-\sin^2 x\right)^k$

Then, the Binomial Theorem allows us to expand $(1-\sin^2 x)^k$:

$$(1-\sin^2 x)^k = 1-k\sin^2 x + \binom{k}{2}\sin^4 x - \cdots + (-1)^k \sin^{2k}(x)$$

Conclude: $\sin^m(x)\cos^n(x) = \sin^m(x)\cos^{2k+1}(x) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\underbrace{\sin^{2j+m}(x)\cos x}_{\substack{\text{easy integration} \\ (u=\sin x \\ \text{substitution!})}}$

- If m is ODD, write $m=2k+1$. We reverse the roles of sin & cos:

$$\sin^{2k+1}(x)\cos^n(x) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\underbrace{\cos^{2j+n}(x)\sin x}_{\text{easy integration by substitution } u=\cos x.}$$

EXAMPLES:

$(m=3,\ n=0)$ $\int \sin^3 x\,dx = \int \sin x\,\sin^2 x\,dx = \int \sin x\,(1-\cos^2 x)\,dx$

$= \int \sin x\,dx - \int \sin x\cos^2 x\,dx \underset{\hookrightarrow u=\cos x}{=} -\cos x + \frac{\cos^3 x}{3} + C$

$(m=2,\ \underline{n=5})$ $\displaystyle\int \sin^2(x)\cos^5 x\,dx = \int \sin^2 x\cos(x)\cos^4(x)\,dx = \int \sin^2 x\cos x\,(1-\sin^2 x)^2\,dx$

$\displaystyle = \int \sin^2 x\cos x\,(1+\sin^4 x - 2\sin^2 x)\,dx = \int \cos x\sin^2 x\,dx + \int \cos x \sin^5 x\,dx$

$\displaystyle -2\int \cos x\sin^4 x\,dx \underset{\hookrightarrow u=\sin x}{=} \frac{\sin^3 x}{3} + \frac{\sin^6 x}{6} - \frac{2}{5}\sin^5 x + C$

$\boxed{\text{CASE B : both } m \,\&\, n \text{ are EVEN (eg } 0,2,4,6\ldots\text{)}}$

Write $m=2k$ & $n=2l$ for $k,l\geq 0$ integers.

Use half-angle formulas!

$$\begin{cases}\cos^2 x + \sin^2 x = 1\\ \cos^2 x - \sin^2 x = \cos(2x)\end{cases} \rightsquigarrow \boxed{\begin{aligned}2\cos^2 x &= 1+\cos 2x & &\text{(Add 2 eqns)}\\ 2\sin^2 x &= 1-\cos 2x & &\text{(Substract 2 eqns)}\end{aligned}}$$

<u>IDEA</u>: $\displaystyle \cos^{2k}(x)\sin^{2l}(x) = \left(\frac{1+\cos 2x}{2}\right)^k\left(\frac{1-\cos 2x}{2}\right)^l$

$\rightsquigarrow$ Expand & use further half-angle formulas until one of the exponents is ODD or 0. Then, we can use CASE A & we get a constant, which we know how to integrate.

<u>EXAMPLES</u>: (1) $\displaystyle\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx \underset{\downarrow\ \tfrac{1}{2}\text{ angle}}{=} \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$\displaystyle = \int \underbrace{\frac{1}{4}}_{\text{constant}\checkmark} + \frac{1}{4}\underbrace{2\cos 2x}_{\text{ODD}\checkmark} + \frac{1}{4}\cos^2 2x\,dx = \frac{1}{4}x + \frac{\sin 2x}{4} + \frac{1}{4}\int \cos^2(2x)\,dx$

$\displaystyle \int \cos^2 2x\,dx \underset{\downarrow\ u=2x}{=} \frac{1}{2}\int \cos^2 u\,du \underset{\downarrow\ \tfrac{1}{2}\text{ angle}}{=} \frac{1}{2}\int\left(\frac{1+\cos 2u}{2}\right)du = \frac{1}{4}\int \underbrace{1}_{\text{const}\checkmark} + \underbrace{\cos 2u}_{\text{ODD}\checkmark}\,du$

$\displaystyle = \frac{1}{4}\left(u + \frac{\sin 2u}{2}\right) = \frac{1}{4}\left(2x + \frac{\sin 4x}{2}\right) = \frac{x}{2} + \frac{\sin 4x}{8}$

<u>Conclude</u> $\displaystyle\int \cos^4 x\,dx = \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{2} + \frac{\sin 4x}{8} + C = \frac{3}{4}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{8} + C$

(2) $\displaystyle\int \cos^2 x\sin^2 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)dx \underset{\downarrow\ u=2x}{=} \frac{1}{4}\int (1+\cos u)(1-\cos u)\frac{du}{2}$

$\displaystyle = \frac{1}{8}\int 1-\cos^2 u\,du = \frac{1}{8}\int 1 - \frac{1+\cos 2u}{2}\,du = \frac{u}{8} - \frac{1}{32}\sin 2u + C = \frac{x}{4} - \frac{\sin(4x)}{32} + C$