Lecture XXXVII: §10.3 (cont) Certain trigonometric integrals
§10.4 Trigonometric substitutions

Recall: Last time, we discussed integration as a recognition problem (what method to use?)

§1. Certain trigonometric examples:

GOAL: Find a method for integrating these 3 functions:

1. $\int \sin^m(x) \cos^n(x)\, dx$
2. $\int \tan^m(x) \sec^n(x)\, dx$ for $m, n \geq 0$ integers.
3. $\int \cot^m(x) \csc^n(x)\, dx$

Last time: we solved 1 depending on parity of m & n (2 cases):

Key: $d(\sin x) = \cos x\, dx$ & $d(\cos x) = -\sin(x)\, dx$ $\leadsto$ need ODD powers!

$\boxed{\text{CASE A: either } m \text{ or } n \text{ are ODD (eg } 1, 3, 5, 7, \ldots)}$

Trick: Use trig identities to turn integrand into sums of functions of the form
(1) $\cos^\ell(x) \sin(x)$ (if m is ODD) or (2) $\sin^\ell(x) \cos(x)$ (if n is ODD)

Why is this good? These terms can be integrated by a simple substitution!

How? We show it for (2) & then for (1):

- If n is ODD, we can write n as $n = 2k+1$ for some integer $k \geq 0$.

Then $\cos^n x = \cos^{2k+1}(x) = \cos(x) \cos^{2k}(x) = \cos(x)\,(\cos^2(x))^k = \cos(x)\,(1-\sin^2 x)^k$

Then, the Binomial Theorem allows us to expand $(1-\sin^2 x)^k$:

$$(1-\sin^2 x)^k = 1 - k \sin^2 x + \binom{k}{2} \sin^4 x - \cdots + (-1)^k \sin^{2k}(x)$$

Conclude:
$$\sin^m(x) \cos^n(x) = \sin^m(x) \cos^{2k+1}(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \underbrace{\sin^{2j+m}(x) \cos x}_{\text{easy integration } (u=\sin x \text{ substitution!})}$$

- If m is ODD, write $m = 2k+1$. We reverse the roles of $\sin$ & $\cos$:

$$\sin^{2k+1}(x) \cos^n(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \underbrace{\cos^{2j+n}(x) \sin x}_{\text{easy integration by substitution } u=\cos x.}$$

$\boxed{\text{CASE B : both } m \ \& \ n \text{ are EVEN } (\text{eg } 0, 2, 4, 6 \ldots)}$

Write $m = 2k$ & $n = 2l$ for $k, l \geq 0$ integers.

Use half-angle formulas!

$$\begin{cases} \cos^2 x + \sin^2 x = 1 \\ \cos^2 x - \sin^2 x = \cos(2x) \end{cases} \rightsquigarrow \boxed{\begin{aligned} 2\cos^2 x &= 1 + \cos 2x \\ 2\sin^2 x &= 1 - \cos 2x \end{aligned}} \quad \begin{aligned} &(\text{Add 2 eqns}) \\ &(\text{Substract 2 eqns}) \end{aligned}$$

IDEA: $\cos^{2k}(x) \sin^{2l}(x) = \left(\frac{1+\cos 2x}{2}\right)^k \left(\frac{1-\cos 2x}{2}\right)^l$

$\rightsquigarrow$ Expand & use further half-angle formulas until one of the exponents is ODD or 0. Then, we can use CASE A & we get a constant, which we know how to integrate.

Q: What about ② $= \int \tan^m x \sec^n x \, dx$ & ③ $= \int \cot^m x \csc^n x \, dx$?

For these 2, we need to treat 3 different cases:

- CASE A = m odd
- CASE B = n even

} overlap (m ODD & n even) $\rightsquigarrow$ can use any of them

- CASE C : neither, so m even & n odd. $\rightsquigarrow$ can't solve (for now!)

We solve the integrals by using specific trigonometric identities & substitutions after expanding $\tan^m x \sec^n x$ or $\cot^m x \csc^n x$ with the Binomial Thm.

② $\begin{cases} \text{CASE A: use } \tan^2 x = \sec^2 x - 1 \quad \& \quad d(\sec x) = \sec x \tan x \, dx \\ \text{CASE B: use } \sec^2 x = 1 + \tan^2 x \quad \& \quad d(\tan x) = \sec^2 x \, dx \end{cases}$

$\left(\frac{1}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\right)$

③ $\begin{cases} \text{CASE A : use } \cot^2 x = \csc^2 x - 1 \quad \& \quad d(\cot x) = -\csc^2 x \, dx \\ \text{CASE B : use } \csc^2 x = 1 + \cot^2 x \quad d(\csc x) = -\cot x \csc x \, dx \end{cases}$

$\left(\frac{1}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right)$

EXAMPLES for ②

$(m = 3, n = 1) \rightsquigarrow$ Case A

$$\int \tan^3 x \sec x\,dx \underset{\text{pull } \tan x \text{ out}}{=} \int \tan(x)\tan^2(x)\sec x\,dx \underset{\text{replace } \tan^2 x}{=} \int \tan x\left(\sec^2 x - 1\right)\sec x\,dx$$

$$\underset{\text{group } (\tan x \sec x\,dx)}{=} \int \sec^2 x \underbrace{\boxed{\tan x \sec x\,dx}}_{=d(\sec x)} - \int \underbrace{\boxed{\tan x \sec x\,dx}}_{=d(\sec x)} = \frac{\sec^3 x}{3} - \frac{\sec^2 x}{2} + C$$

Steps: 1. $\tan^{2k+1}(x)\sec(x) = \tan x \sec x\left(\sec^2 x - 1\right)^k$

2. Expand $\left(\sec^2 x - 1\right)^k$ with Binomial Theorem

3. Use substitution $u = \sec x$ $\quad du = \tan x \sec x\,dx$

$(m=4,\ \underline{n=6}) \rightsquigarrow$ Case B

$$\int \tan^4 x \sec^6 x\,dx \underset{\text{pull } \sec^2 x \text{ out}}{=} \int \tan^4 x \sec^4 x \sec^2 x\,dx \underset{\text{replace } \sec^4 x}{=} \int \tan^4 x\left(1+\tan^2 x\right)^2 \sec^2 x\,dx$$

$$\underset{\text{expand}}{=} \int \tan^4 x\left(1 + \tan^4 x + 2\tan^2 x\right)\sec^2 x\,dx$$

$$\underset{\text{group } \sec^2 x\,dx}{=} \int \tan^4 x \underbrace{\sec^2 x\,dx}_{=d\tan(x)} + \int \tan^8 x \underbrace{\sec^2 x\,dx}_{=d(\tan x)} + 2\int \tan^6 x \underbrace{\sec^2 x\,dx}_{=d(\tan x)}$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^9 x}{9} + \frac{2}{7}\tan^7 x + C$$

Steps: 1. $\tan^m(x)\sec^{2k} x\,dx = \tan^m x\left(\sec^2 x\right)^{k-1} x \sec^2 x\,dx$

2. Write $\left(\sec^2 x\right)^{k-1} = \left(1+\tan^2 x\right)^{k-1}$ & expand with Binomial Theorem

3. Use substitution $u = \tan x$ $\quad du = \sec^2 x\,dx$.

⚠ Case C we cannot yet solve, with one exception $m=0$ & $n=1$.

- $\int \sec u\,du = \ln\left(\sec(u) + \tan(u)\right) + B$
- $\int \csc u\,du = -\ln\left(\csc(u) + \cot(u)\right) + B$

(B constant.)

Remark: Sometimes it's easier to express $\tan^m x \sec^n x$ or $\cot^m(x)\csc^n x$ using $\sin x$ & $\cos x$. Issue: exponents can be negative!

Example:
$$\int \tan x \sec x\,dx = \int \frac{\sin x}{\cos x}\,\frac{1}{\cos x}\,dx = \int \frac{\sin x}{\cos^2 x}\,dx$$

$$\underset{u=\cos x}{=} \int \frac{-du}{u^2} = \frac{1}{u} + C = \frac{1}{\cos x} + C = \sec x + C$$

We already know this because $d\sec x = \tan x \sec x\,dx$

§2 Trigonometric Substitutions

This method is used to compute integrals involving either of these 3 expressions: (1) $\sqrt{a^2-x^2}$ where $a>0$ is a constant.
(2) $\sqrt{a^2+x^2}$
(3) $\sqrt{x^2-a^2}$

Each expression requires a different substitution of x as a trig function

(1) $x = a\sin u \rightsquigarrow \sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2 u} = \sqrt{a^2\cos^2 u} = a\cos u$

use $1 = \cos^2 u + \sin^2 u$

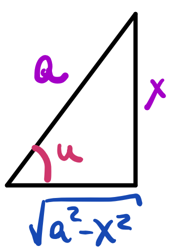
(2) $x = a\tan u \rightsquigarrow \sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2 u} = \sqrt{a^2\sec^2 u} = a\sec u$

use $1+\frac{\sin^2 u}{\cos^2 u} = \frac{1}{\cos^2 u} = \sec^2 u$

(3) $x = a\sec u \rightsquigarrow \sqrt{x^2-a^2} = \sqrt{a^2\sec^2 u - a^2} = \sqrt{a^2\tan^2 u} = a\tan u$

Geometric Interpretation: x, a are sides of a right triangle.

(1) $\sqrt{a^2-x^2}$

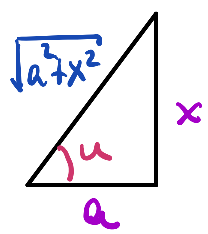


$x = a\sin u$

$\cos u = \frac{\sqrt{a^2-x^2}}{a}$

$\tan u = \frac{x}{\sqrt{a^2-x^2}}$

(2) $\sqrt{a^2+x^2}$

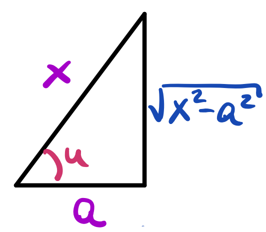


$x = a\tan u$

$\cos u = \frac{a}{\sqrt{a^2+x^2}}$

$\sin u = \frac{x}{\sqrt{a^2+x^2}}$

(3) $\sqrt{x^2-a^2}$



$a = x\cos u$

$\rightsquigarrow x = \frac{a}{\cos u} = a\sec u$

$\sin u = \frac{\sqrt{x^2-a^2}}{x}$

$\tan u = \frac{\sqrt{x^2-a^2}}{a}$

Examples

① $\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \int \frac{a\cos u}{a\sin u}\, a\cos u\, du = a\int \frac{\cos^2 u}{\sin u}\,du$

$x = a\sin u \quad dx = a\cos u\,du$

$= a\int \frac{1-\sin^2 u}{\sin u}\,du = a\int \frac{1}{\sin u}\,du - a\int \sin u\,du.$

$= -a\ln(\csc(u)+\cot(u)) + a\cos u + C$

TABLE

In terms of x? • $\csc(u) = \frac{1}{\sin u} = \frac{a}{x}$,

• $\cos u = \sqrt{1-\sin^2 u} = \sqrt{1 - x^2/a^2} = \frac{\sqrt{a^2-x^2}}{a}$,

• $\cot(u) = \frac{\cos u}{\sin u} = \frac{(\sqrt{a^2-x^2})/a}{x/a} = \frac{\sqrt{a^2-x^2}}{x}$

Conclude $$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = -a\ln\left(\frac{a}{x}+\frac{\sqrt{a^2-x^2}}{x}\right) + \sqrt{a^2-x^2} + C$$

(2) $$\int \frac{dx}{x^2\sqrt{a^2+x^2}} \underset{\substack{x=a\tan u\\ dx = a\sec^2 u\,du}}{=} \int \frac{a\sec^2 u\,du}{a^2\tan u\; a\sec u} = \frac{1}{a^2}\int\frac{\sec u}{\tan^2 u}\,du = \frac{1}{a^2}\int \frac{\cos^2 u}{\sin^2 u}\,\frac{du}{\cos u}$$

$$= \frac{1}{a^2}\int\frac{\cos u\,du}{\sin^2 u} \underset{y=\sin u \quad dy=\cos u\,du}{=} \frac{1}{a^2}\int\frac{dy}{y^2} = \frac{-1}{a^2 y}+C = \frac{-1}{a^2\sin u}+C$$

Again, we need to write $\sin u$ in terms of x!

$a\sec u = \sqrt{a^2+x^2}$ gives $\cos u = \frac{a}{\sqrt{a^2+x^2}}$

So $\sin u = \cos u\tan u = \frac{a}{\sqrt{a^2+x^2}}\,\frac{x}{a} = \frac{x}{\sqrt{a^2+x^2}}$.

Conclude: $$\int\frac{dx}{x^2\sqrt{a^2+x^2}} = \frac{-\sqrt{a^2+x^2}}{a^2\, x}+C$$

We can verify this! $$\frac{d}{dx}\left(\frac{-\sqrt{a^2+x^2}}{a^2\, x}\right) = \left(\frac{-x a^2 x}{\sqrt{a^2+x^2}} + a^2\sqrt{a^2+x^2}\right)\Big/a^4x^2 = \frac{-a^2x^2+a^2(a^2+x^2)}{a^4x^2\sqrt{a^2+x^2}} = \frac{1}{x^2\sqrt{a^2+x^2}} \checkmark$$

(3) $$\int\frac{\sqrt{x^2-a^2}}{x}\,dx \underset{\substack{x=a\sec u\\ dx=a\sec u\tan u\,du}}{=} \int\frac{a\tan u}{a\sec u}\,a\sec u\tan u\,du = \int a\tan^2 u\,du$$

$$\underset{\substack{\text{part §1}\\ n=2 \text{ EVEN}}}{=} a\int(\sec^2u-1)\,du = a\int \underbrace{\sec^2 u\,du}_{=d(\tan u)} - a\int du = a\tan u - au + C$$

Again, we need to write u & tan u in terms of x! $a\tan u = \sqrt{x^2-a^2}$

$u = \arctan\left(\frac{\sqrt{x^2-a^2}}{a}\right)$

Conclude: $$\int\frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\arctan\left(\frac{\sqrt{x^2-a^2}}{a}\right) + C$$