Lecture XXXVIII: § 10.5 Completing the square
§ 10.6 Partial Fractions

Last Time: Used trig substitutions to integrate expressions with $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$
$= a\cos u$ | $= a/\cos u$ | $= a\tan u$
$x = a\sin u$ | $x = a\tan u$ | $x = a\sec u$

§1 Completing the square:

GOAL: Integrate expressions involving
(1) ax^2+bx+c or (2) $\sqrt{ax^2+bx+c}$
where a, b, c are fixed constants & $a \neq 0$.

To use trigonometric substitutions we need to write the quadratic expression as $\pm(A^2 \pm u^2)$

IDEA: Reverse the process of squaring a sum:
$$(x+A)^2 = x^2 + (2A)x + A^2$$

Example 1: add & substract $\left(-\frac{4}{2}\right)^2 = 4$
$$5+4x-x^2 = -(x^2-4x-5) = -(x^2-4x+4-4-5)$$
$$= -((x-2)^2-9) = 9-(x-2)^2$$

So
$$\int \frac{x+2}{\sqrt{5+4x-x^2}}\,dx = \int \frac{x+2}{\sqrt{9-(x-2)^2}}\,dx = \int \frac{u+4}{\sqrt{9-u^2}}\,du$$
$u = x-2$, $du = dx$

Now: use trig substitution $u = 3\sin y$ $(a=3)$
$$= \int \frac{3\sin y+4}{3\cos y}\,3\cos y\,dy = \int 3\sin y + 4\,dy = -3\cos y + 4y + C$$
$$= -\sqrt{9-u^2} + 4\arcsin\frac{u}{3} + C = -\sqrt{9-(x-2)^2} + 4\arcsin\left(\frac{x-2}{3}\right) + C$$

Example 2:
$$x^2+2x+10 = x^2+2x+1-1+10 = (x+1)^2+9$$
$$\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{9+(x+1)^2} = \int \frac{du}{9+u^2} = \frac{1}{9}\int \frac{du}{1+\left(\frac{u}{3}\right)^2} = \frac{1}{3}\int \frac{dv}{1+v^2}$$
$u = x+1$, $v = \frac{u}{3}$
$$= \frac{1}{3}\arctan v + C = \frac{1}{3}\arctan(3u) + C = \frac{1}{3}\arctan(3x+3) + C$$

Example 3
$$x^2-2x+5 = x-2x+1-1+5 = (x-1)^2+4$$

$$\int \frac{x\,dx}{\sqrt{x^2-2x+5}} = \int \frac{x\,dx}{\sqrt{4+(x-1)^2}} \underset{u=x-1}{=} \int \frac{u+1}{\sqrt{4+u^2}}\,du = \int \frac{u}{\sqrt{4+u^2}}\,du + \int \frac{du}{\sqrt{4+u^2}}$$

- For the first integral: use $v = 4+u^2$ $\quad dv = 2u\,du$

$$\int \frac{u}{\sqrt{4+u^2}}\,du = \frac{1}{2}\int \frac{dv}{\sqrt{v}} = \frac{1}{2}\, 2\, v^{1/2} = \sqrt{4+u^2}$$

- For the second integral: use trig substitution $u = 2\tan y$

$$\int \frac{du}{\sqrt{4+u^2}} = \int \frac{1}{2/\cos y}\, 2\sec^2 y\, dy = \int \cos y \frac{1}{\cos^2 y}\, dy = \int \frac{1}{\cos y}\, dy$$

$$= \int \sec y\, dy = \ln(\sec y + \tan y) = \ln\left(\frac{\sqrt{4+u^2}}{2} + \frac{u}{2}\right)$$

Conclude: $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-2x+5}} = \sqrt{4+u^2} + \ln\left(\frac{\sqrt{4+u^2}+u}{2}\right) + C$

$$= \sqrt{4+(x-1)^2} + \ln\left(\frac{\sqrt{4+(x-1)^2} + x-1}{2}\right) + C$$

Example 4 $\quad x^2-4x+3 = x^2-4x+4-4+3 = (x-2)^2-1$

$$\int \frac{dx}{\sqrt{x^2-4x+3}} = \int \frac{dx}{\sqrt{(x-2)^2-1}} \underset{u=x-2}{=} \int \frac{du}{\sqrt{u^2-1}} \underset{\text{trig subst } u=\sec y}{=} \int \frac{\sec y \tan y}{\tan y}\, dy = \int \sec y\, dy$$

$$= \ln(\sec y + \tan y) + C = \ln\left(u + \sqrt{u^2-1}\right) + C = \ln\left((x-2) + \sqrt{(x-2)^2+1}\right) + C$$

In general: $a \neq 0$

$$ax^2+bx+c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right)$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

$$= a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a}$$

The sign of b^2-4ac determines which trigonometric substitution must be used to integrate.

§ 2 Partial Fractions

GOAL Integrate rational functions = ratios of polynomials $\frac{P(x)}{Q(x)}$

STEP 1: Reduce to the case where $\deg P < \deg Q$.

For this, we will need to review how long division works.

Long division: **Input** $P_{(x)}, Q_{(x)}$

Output $S(x)$ = quotient & $R_{(x)}$ = remainder

satisfying $P_{(x)} = S_{(x)} Q_{(x)} + R_{(x)}$

with $R = 0$ or $\deg R < \deg Q$.

Example:

$$\begin{array}{rl} P = & x^3 - 3x^2 \quad \big|\ x^2+1 = Q \\ - & -x^3 \qquad + x \quad \boxed{x-3} = S \\ \hline & -3x^2 - x \\ - & -3x^2 \qquad -3 \\ \hline & \boxed{-x+3} = R \end{array}$$

(degree $= 1 < 2 =$ degree Q.)

Routine

1. Take leading terms (highest degree monomial & coefficient) & write ratio ($x = \frac{x^3}{x^2}$, $-3 = \frac{-3x^2}{x^2}$)
2. Multiply ratio by Q & substract it from P
3. Repeat with new polynomial until we get 0 or degree $< \deg Q$.
4. Left over is the remainder R. Sum of all ratios is S.

So $x^3 - 3x^2 = \underbrace{(x-3)}_{S(x)}(x^2+1) + \underbrace{(-x+3)}_{R(x)}$

Example 2:

$$\begin{array}{rl} & x^5 + 2x + 1 \quad \big|\ x^3 - 2 \\ - & x^5 - 2x^2 \qquad\quad x^2 \\ \hline & 2x^2 + 2x + 1 \ \checkmark \end{array}$$

$\rightsquigarrow x^5 + 2x + 1 = x^2(x^3-2) + (2x^2+2x+1)$

Conclusion:

$$\int \frac{P}{Q}\,dx = \int \frac{SQ+R}{Q}\,dx = \int S_{(x)}\,dx + \int \frac{R_{(x)}}{Q_{(x)}}\,dx$$

& $R_{(x)} = 0$ or $\deg R < \deg Q$.

↳ only case we need to analyze.

CRUCIAL FACT: Every polynomial with real coefficient can be written as a product of linear (deg 1) or irreducible degree 2 polynomials ↪ no real roots, like x^2+1.

These deg 2 polynomials have the form $a(x^2+bx+c)$ with $b^2-4c<0$ (roots $= \frac{-b\pm\sqrt{b^2-4c}}{2}$ are not real!)

This fact will allow us to simplify our task:

STEP 2: Treat 2 cases

① $\begin{cases} Q = (x-\lambda)^m \\ P = 1 \end{cases}$ ② $\begin{cases} Q = (x^2+bx+c)^m \\ (\text{with } b^2-4c<0) \\ P = a_1x+a_0 \quad (a_1, a_0 \text{ constants}) \end{cases}$

Each case has its own technique:

① $$\int \frac{1}{(x-\lambda)^m}\,dx \underset{\substack{u=x-\lambda \\ du=dx}}{=} \int \frac{du}{u^m} = \begin{cases} \ln u + C & \text{if } m=1 \\ \frac{u^{-m+1}}{-m+1} + C & \text{if } m\neq 1 \end{cases}$$

② $$\int \frac{x+}{(x^2+bx+c)^m}\,dx$$ need to complete the square!

$$x^2+bx+c = x+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c = \left(x+\frac{b}{2}\right)^2-\underbrace{\left(\frac{b^2-4c}{4}\right)}_{<0}$$

Call $\frac{b^2-4c}{4} = -A^2$ for some $A>0$

$$x^2+bx+c = \left(x+\frac{b}{2}\right)^2+A^2$$

so $$\int \frac{a_1x+a_0}{(x^2+bx+c)^m}\,dx = \int \frac{a_1x+a_0}{\left(\left(x+\frac{b}{2}\right)^2+A^2\right)^m}\,dx \underset{u=x+\frac{b}{2}}{=} \int \frac{a_1(u-b/2)+a_0}{(u^2+A^2)^m}\,du$$

$$= \underbrace{\boxed{a_1}}_{\text{constant}} \int \frac{u\,du}{(u^2+A^2)^m} + \underbrace{\boxed{\left(a_0-a_1\frac{b}{2}\right)}}_{\text{constant}} \int \frac{du}{(u^2+A^2)^m}$$

. For the first integral: $v = u^2+A^2$ $dv = 2u\,du$

$$\int \frac{u\,du}{(u^2+A^2)^m} = \int \frac{dv}{2v^m} = \begin{cases} \frac{1}{2}\ln|v| = \frac{1}{2}\ln|u^2+A^2| & \text{if } m=1 \\ \frac{1}{2}\frac{v^{-m+1}}{-m+1} = \frac{(u^2+A^2)^{1-m}}{2(1-m)} & \text{if } m\neq 1 \end{cases}$$

- For the second integral: use trig substitution $u = A\tan v$
  $du = A\sec^2 v\, dv$

$$\int \frac{du}{(u^2+A^2)^m} = \int \frac{A\sec^2 v\, dv}{(A/\cos v)^{2m}} = \frac{1}{A^{2m-1}} \int (\cos v)^{2m-2}\, dv$$

$$= \int \frac{1}{A^{2m-1}} \left(\frac{1+\cos 2v}{2}\right)^{m-1} dv$$

↓ ½ angle for cos.
$2m-2 = 2(m-1)$

and keep going, as we did in Lecture 36 (§10.3)

STEP 3: Use these 3 cases to solve $\int \frac{P}{Q}\,dx$ when $\deg P < \deg Q$.

This requires the use of Partial Fractions (next Time!)

We will write any $\frac{P}{Q}$ as a sum of rational functions of the form $\frac{a_1}{(x-\lambda)^m}$ and/or $\frac{a_1 x + a_0}{(x^2+bx+c)^m}$ with $b^2-4c<0$

for various powers $m>0$.