Lecture XXXIX: § 10.6 (cont.) Partial Fractions

GOAL Integrate rational functions = ratios of polynomials $\frac{P(x)}{Q(x)}$ $Q \neq 0$

- **STEP 1**: Reduce to the case where $Q = 1$ or $\deg P < \deg Q$ (by long division)

Long division gives $S(x), R(x)$ polynomials with $P_{(x)} = S_{(x)} Q_{(x)} + R_{(x)}$ and

$R = 0$ or $\deg R < \deg Q$

$$\rightsquigarrow \frac{P}{Q} = \underset{\substack{\uparrow \\ \text{easy to integrate}}}{S} + \frac{R}{Q}$$

- **CRUCIAL FACT**: Every polynomial with real coefficient can be written as a product of linear (deg 1) or irreducible degree 2 polynomials

↳ no real roots, like x^2+1.

These deg 2 polynomials have the form $a(x^2+bx+c)$ with $b^2-4c<0$

$\left(\text{roots} = \frac{-b \pm \sqrt{b^2-4c}}{2} \text{ are not real!}\right)$

This fact will allow us to simplify our task:

STEP 2: Treat 2 cases

① $\begin{cases} Q = (x-\lambda)^m \\ P = 1 \end{cases}$

$\wr$ substitute $u = x-\lambda$.

② $\begin{cases} Q = (x^2+bx+c)^m \\ (\text{with } b^2-4c<0) \\ P = a_1 x + a_0 \quad (a_1, a_0 \text{ constants}) \end{cases}$

$\rightsquigarrow$ complete the square to get $Q_{(x)} = ((x-\lambda)^2 + A^2)^m$ & substitute $u = x-\lambda$

↑ HARDER ONE

STEP 3: Use these 2 cases to solve $\int \frac{P}{Q}\,dx$ when $\deg P < \deg Q$.

This requires the use of Partial Fractions

IDEA: Write $\frac{P}{Q}$ with $\deg P < \deg Q$ as a sum of rational functions of the form $\frac{A}{(x-\lambda)^m}$ or $\frac{Ax+B}{(x^2+bx+c)^r}$, with $m, r > 0$ & $b^2-4c<0$

How? Reverse the process of written common denominator (= factorized Q)

EXAMPLE 1: $\frac{12x-7}{(x-1)(x-2)} = \frac{-5}{x-1} + \frac{17}{x-2}$ (-5 & 17 must be determined!)

So $\int \frac{12x-7}{(x-1)(x-2)}\,dx = \int \frac{-5}{x-1}\,dx + \int \frac{17}{x-2}\,dx = -5\ln|x-1| + 17\ln|x-2| + C$

The construction depends on the factors of Q and their multiplicity:

CASE 1: $Q_{(x)} = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$ Q has all real roots & all are different

- We write $\frac{P(x)}{Q(x)} = \frac{A_1}{x-\lambda_1} + \frac{A_2}{x-\lambda_2} + \cdots + \frac{A_n}{x-\lambda_n}$

- We can determine $A_1, \dots, A_n$ in 3 ways:

① Take common denominator & equate each coefficient in $P_{(x)}$ with (RHS) numerator

② Evaluate numerator in (RHS) at the n roots to find $A_1, .. A_n$

③ Without taking common denominator: evaluate at n random numbers to get n equations in n unknowns $A_1, \dots, A_n$

EXAMPLE 2: $\frac{9x^2+6}{x(x-2)(x-3)} = \frac{A_1}{x} + \frac{A_2}{x-2} + \frac{A_3}{x-3}$

Method ① Common Denominator

$$\text{Numerator} = A_1(x-2)(x-3) + A_2\,x(x-3) + A_3\,x(x-2)$$
$$= A_1(x^2-5x+6) + A_2(x^2-3x) + A_3(x^2-2x)$$
$$= (A_1+A_2+A_3)x^2 + (-5A_1-3A_2-2A_3)x + 6A_1$$

Equate with P: $\begin{cases} 9 = A_1+A_2+A_3 & \text{coeff } x^2 \\ 0 = -5A_1-3A_2-2A_3 & \text{coeff } x \\ 6 = 6A_1 & \text{coeff } 1 \quad \rightsquigarrow A_1 = 1 \end{cases}$

Replace A_1 to get $\begin{cases} 9 = 1+A_2+A_3 \\ 0 = -5-3A_2-2A_3 \end{cases} \rightsquigarrow \begin{matrix} 8 = A_2+A_3 \rightsquigarrow A_3 = 8-A_2 \\ 5 = -3A_2-2A_3 \end{matrix}$

Replace A_3 in last equation: $5 = -3A_2 - 2(8-A_2) = -A_2 - 16 \rightsquigarrow A_2 = -21$

So $A_3 = 8 - A_2 = 8-(-21) = 29$

Conclusion: $A_1 = 1$, $A_2 = -21$ & $A_3 = 29$

Method ② $9x^2+6 = A_1(x-2)(x-3) + A_2\,x(x-3) + A_3\,x(x-2)$

At $x=0 \rightsquigarrow 6 = A_1(-2)(-3) + 0 + 0 = 6A_1 \rightsquigarrow A_1 = 1$

At $x=2 \rightsquigarrow 42 = 0 + A_2\cdot 2(2-3) + 0 = -2A_2 \rightsquigarrow A_2 = -21$

At $x=3 \rightsquigarrow 87 = 0 + 0 + A_3\cdot 3\cdot 1 = 3A_3 \rightsquigarrow A_3 = 29$

Method ③ At $x=1$: $\frac{P}{Q} = \frac{15}{1(-1)(-2)} = A_1 - A_2 - \frac{A_3}{2}$

At $x=-1$: $\frac{P}{Q} = \frac{15}{-1(-3)(-4)} = -A_1 - \frac{A_2}{3} - \frac{A_3}{4}$

At $x=-2$: $\frac{P}{Q} = \frac{42}{(-2)(-4)(-5)} = \frac{A_1}{-2} - \frac{A_2}{4} - \frac{A_3}{5}$

$$\rightsquigarrow \begin{cases} 15 = 2A_1 - 2A_2 - A_3 \\ 15 = 12A_1 + 4A_2 + 3A_3 \\ 42 = 20A_1 + 10A_2 + 8A_3 \end{cases}$$ solution is the same as before.

CASE 2: Q has all real roots but some have multiplicity ≥ 2.

Write $Q(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots\cdots(x-\lambda_n)^{m_n}$ with $m_1, \ldots m_n \geq 1$. (roots are distinct!)

Solution: Replace $\frac{A_k}{x-\lambda_k}$ from CASE 1 by the sum

$$\frac{A_{k,1}}{(x-\lambda_k)} + \frac{A_{k,2}}{(x-\lambda_k)^2} + \cdots\cdots + \frac{A_{k,m_k}}{(x-\lambda_k)^{m_k}}$$ with $m_k = \text{mult}(\lambda_k, Q)$

So the # of terms = multiplicity of λ_k are a root of Q.

→ We find all coefficients $A_{i,j}$ as we did in CASE 1. (for Method ② we will also need to compute derivatives up to order m_k-1 & evaluate them at $x=\lambda_k$).

EXAMPLE 3: $\frac{2x+1}{(x-1)^3} = \frac{A_{1,1}}{(x-1)} + \frac{A_{1,2}}{(x-1)^2} + \frac{A_{1,3}}{(x-1)^3}$

Numerator: $2x+1 = A_{1,1}(x-1)^2 + A_{1,2}(x-1) + A_{1,3}$

- Evaluate at $x=1$: $3 = 0 + 0 + A_{1,3} \rightsquigarrow A_{1,3} = 3$
- Take derivatives up to order 2 & evaluate at $x=1$

First derivative: $2 = 2A_{1,1}(x-1) + A_{1,2}$

At $x=1$: $2 = 0 + A_{1,2} \rightsquigarrow A_{1,2} = 2$

Second derivative: $0 = 2A_{1,1} \rightsquigarrow A_{1,1} = 0$

Conclude: $A_{1,1} = 0$, $A_{1,2} = 2$, $A_{1,3} = 3$.

EXAMPLE 4: $\dfrac{1}{(x-1)(x-2)^2} = \dfrac{A_1}{x-1} + \dfrac{A_{2,1}}{(x-2)} + \dfrac{A_{2,2}}{(x-2)^2}$

Numerator: $1 = A_1(x-2)^2 + A_{2,1}(x-1)(x-2) + A_{2,2}(x-1)$

At $x=1$ $\quad 1 = A_1(-1)^2 + 0 + 0 \rightsquigarrow A_1 = 1$

At $x=2$: $\quad 1 = 0 + 0 + A_{2,2}(2-1) \rightsquigarrow A_{2,2} = 1$

- Derivative: $0 = 2A_1(x-2) + A_{2,1}\left((x-2)+(x-1)\right) + A_{2,2}$

At $x=2$: $\quad 0 = 0 + A_{2,1}(2-1) + A_{2,2} = A_{2,1}+1$

$\rightsquigarrow A_{2,1} = -1$

Conclude:

$$\int \frac{dx}{(x-1)(x-2)^2} = \int \frac{1}{x-1}\,dx + \int \frac{-1}{x-2}\,dx + \int \frac{1}{(x-2)^2}\,dx$$

$$= \ln|x-1| - \ln|x-2| - \frac{1}{(x-2)} + C$$

$\boxed{\text{CASE 3:}}$ Q has quadratic factors with no real roots. ($= x^2+bx+c$ with $b^2-4c<0$)

The decomposition depends on the multiplicity of the factor.

- If mult $=1$, then we have a summand $\dfrac{Ax+B}{x^2+bx+c}$ in the partial fraction decomposition.

- If mult $= m > 1$, then we have m summands in the partial faction decomposition, namely:

$$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots\cdots + \frac{A_mx+B_n}{(x^2+bx+c)^m}$$

We find the values of (A,B) & $(A_1,B_1,\dots,A_m,B_m)$ with the same methods used for Cases 1 & 2.

EXAMPLE 5: $\dfrac{x^2+2}{x^4+2x^3+2x^2} = \dfrac{x^2+2}{x^2(x^2+2x+2)}$

$2^2-4\cdot 2<0$ so x^2+2x+2 has no real roots.

Write $\dfrac{x^2+2}{x^4+2x^3+2x^2} = \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \dfrac{A_3x+B_3}{x^2+2x+2}$

Numerator: $x^2+2 = A_1x(x^2+2x+2) + A_2(x^2+2x+2) + (A_3x+B_3)x^2$

- Evaluate at $x=0$: $2 = 0 + A_2\cdot 2 \rightsquigarrow A_2=1$
- Derivative: $2x = A_1\left(x^2+2x+2 + x(2x+2)\right) + A_2(2x+2) +$

  $+(A_3\, 3x^2 + 2B_3x)$

  At $x=0$: $0 = A_1(2+0) + A_2\, 2 + (A_3\cdot 0+B_3\cdot 0)$

  $0 = 2A_1 + 2A_2 = 2A_1+2 \rightsquigarrow A_1=-1$
- Evaluate at 2 random numbers to get equations for A_3 & B_3.

  At $x=1$ $\quad 3 = (-1)\,1\,(1+2+2) + 1\cdot 5 + (A_3+B_3)$

  $3 = A_3+B_3$

  At $x=-1$ $\quad 3 = (-1)(-1)(1-2+2) + 1(1) + (-A_3+B_3)$

  $3-2 = -A_3+B_3$

$\rightsquigarrow \begin{cases} 3 = A_3+B_3 \\ 1 = -A_3+B_3 \end{cases}$ sum $= 4 = 2B_3 \rightsquigarrow B_3=2$; difference: $2 = 2A_3 \rightsquigarrow A_3=1$

Conclusion: $\dfrac{x^2+2}{x^4+2x^3+2x^2} = \dfrac{-1}{x} + \dfrac{1}{x^2} + \dfrac{x+2}{x^2+2x+2}$

$$\int \frac{x^2+2}{x^4+2x^3+2x^2}\,dx = \int -\frac{dx}{x} + \int \frac{dx}{x^2} + \int \frac{x+2}{x^2+2x+2}\,dx$$

$$\int \frac{x+2}{x^2+2x+2}\,dx \underset{\substack{\uparrow \\ \text{Complete the square}}}{=} \int \frac{x+2}{(x+1)^2-1+2}\,dx = \int \frac{x+2}{(x+1)^2+1}\,dx \underset{\substack{\downarrow \\ u=x+1 \\ du=dx}}{=} \int \underset{x+2=u+1}{\frac{u+1}{1+u^2}\,du}$$

$$= \int \frac{u}{1+u^2}\,du + \int \frac{du}{1+u^2} = \frac{1}{2}\ln|1+u^2| + \arctan u$$

So $\displaystyle\int \frac{x^2+2}{x^4+2x^3+2x^2}\,dx = -\ln|x| - \frac{1}{x} + \frac{1}{2}\ln|1+(x+1)^2| + \arctan(1+x) + C$