Lecture XL: §10.7 Integration by parts
§10.8 A Mixed Bag: miscellaneous Types

§1. Integration by parts:

IDEA: Reverse product rule

$d(uv) = u\,dv + v\,du \rightsquigarrow uv = \int d(uv) = \int u\,dv + \int v\,du$

Write $\underbrace{\int u\,dv}_{\text{HARD TO DO}} = uv - \underbrace{\int v\,du}_{\text{EASIER TO DO}}$ & assume we can compute $\int v\,du$

EXAMPLES: ① $\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = x e^x - \int e^x dx = xe^x - e^x + C$

$v = \int e^x dx = e^x$

② $\int \underbrace{\ln x}_{=u} \underbrace{dx}_{=dv} = x \ln x - \int \underbrace{\frac{1}{x}}_{=du} \underbrace{x}_{v}\, dx = x \ln x - x + C.$

$v = \int dx = x$

③ $\int \underbrace{(\ln x)^2}_{=u} \underbrace{dx}_{dv} = (\ln x)^2 x - \int \underbrace{2 \ln x \frac{1}{x}}_{=du} x\, dx = x(\ln x)^2 - 2 \underbrace{\int \ln x\, dx}_{=②}$

$= x(\ln x)^2 - 2x \ln x + 2x + C.$

§2. Fun Recursions:

Integration by parts can lead to interesting recursive formulas

EXAMPLE 1: Define $I_n := \int x^n e^x dx$ for $n \geq 0$ integer.

- Easy base case: $n=0$ $\quad I_0 = \int \underbrace{x^0}_{=1} e^x dx = \int e^x dx = e^x + C$
- Recursion: write I_n in terms of $I_{n-1}, I_{n-2}, \ldots, I_1, I_0$

  (We'll get this from integration by parts)

$I_n = \int \underbrace{x^n}_{=u} \underbrace{e^x dx}_{=dv} = x^n e^x - \int \underbrace{n x^{n-1}}_{du} \underbrace{e^x}_{v} dx = x^n e^x - n I_{n-1}$

$\rightsquigarrow v = \int e^x dx = e^x$

Repeat: $I_{n-1} = x^{n-1}e^x - (n-1)I_{n-2} \rightsquigarrow I_n = x^n e^x - n x^{n-1}e^x + n(n-1)I_{n-2}$

Once more: $I_{n-2} = x^{n-2}e^x - (n-2)I_{n-3}$

$$\rightsquigarrow I_n = x^n e^x \underbrace{-n}_{=(-1)^1\frac{n!}{(n-1)!}} x^{n-1}e^x + \underbrace{n(n-1)}_{=(-1)^2\frac{n!}{(n-2)!}} x^{n-2}e^x \underbrace{-n(n-1)(n-2)}_{=(-1)^3\frac{n!}{(n-3)!}} I_{n-3}$$

Guess: $I_n = \sum_{k=0}^{n-1} (-1)^k \frac{n!}{(n-k)!} x^{n-k} e^x + (-1)^n n! \underbrace{I_0}_{e^x + C}$

$$= \sum_{k=0}^{n} (-1)^n \frac{n!}{(n-k)!} x^{n-k} e^x + C$$

EXAMPLE 2: Define $J_p := \int \sin^p\theta \, d\theta$ for $p \geq 1$

- Base case: $p=1$ $J_1 = \int \sin\theta \, d\theta = -\cos\theta + C$
- Recursion: integration by parts for $p \geq 2$.

$$J_p = \int \sin^p\theta \, d\theta \underset{\text{borrow } \sin\theta}{=} \int \underbrace{\sin^{p-1}\theta}_{u} \underbrace{\sin\theta \, d\theta}_{dv} \qquad v = -\cos v$$

$$= -\sin^{p-1}\theta\cos\theta + \int \cos\theta\,(p-1)\sin^{p-2}\theta\cos\theta \, d\theta$$

$$= -\sin^{p-1}\theta\cos\theta + (p-1)\int \sin^{p-2}\underbrace{\cos^2\theta}_{=1-\sin^2\theta} d\theta$$

$$= -\sin^{p-1}\theta\cos\theta + (p-1)\underbrace{\int \sin^{p-2}\theta \, d\theta}_{=: J_{p-2}} - (p-1)\underbrace{\int \sin^{p-2+2}\theta \, d\theta}_{= J_p}$$

So $p J_p = -\sin^{p-1}\theta\cos\theta + (p-1)J_{p-2}$

$$\boxed{J_p = -\frac{\sin^{p-1}\theta\cos\theta}{p} + \frac{p-1}{p} J_{p-2}}$$

Note p & $p-2$ have the same parity!

This recursion says $\boxed{1} \to 3 \to 5 \to 7 \to \cdots$
$\boxed{2} \to 4 \to 6 \to 8 \to \cdots$

We need to determine $p=2$ case: (it's also a base case)

$$J_2 = \int \sin^2\theta \, d\theta \underset{\frac{1}{2}\text{ angle}}{=} \int \frac{1-\cos 2\theta}{2} d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C = \frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2} + C$$

Observation: For p odd, we can use Trigonometric integration methods from Lectures 36 & 37 to get a close formula for J_p when p is odd (1,3,5,...)

Indeed $J_{2k+1} = \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{j+1} \cos^{2j+1}\theta}{2j+1} + C$

Why? $J_{2k+1} = \int \sin^{2k+1}\theta \, d\theta = \int \underbrace{\sin^{2k}\theta}_{(\sin^2\theta)^k} \, \underbrace{\sin\theta \, d\theta}_{d(-\cos\theta)} = -\int (1-\cos^2\theta)^k d(\cos\theta)$

$\underset{u=\cos\theta}{=} \int (1-u^2)^k du \underset{\text{Binomial Thm}}{=} \int \sum_{j=0}^{k} (-1)^j \binom{k}{j} u^{2j} du$

$\underset{\text{swap sum } \& \int}{=} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \int u^{2j} du = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{u^{2j+1}}{2j+1} + C$

$\underset{u=\cos\theta}{=} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\cos^{2j+1}\theta}{2j+1} + C$

§3. Integration techniques:

GOAL: Reduce calculations to 15 fundamental formulas on the handout

We can use 5 methods:

(1) Substitutions

(2) Trig Substitutions $\begin{cases} x = a\sin\theta & \text{for } a^2-x^2 \\ x = a\tan\theta & \text{for } a^2+x^2 \\ x = a\sec\theta & \text{for } x^2-a^2 \end{cases}$

(3) Partial fractions & completing the square

(4) Integration by parts

(5) Trig identities + simplifications

EXAMPLES ① $\int \frac{dx}{x^2-a^2}$ ($a \neq 0$) $= \int \frac{dx}{(x-a)(x+a)} = \frac{1}{2a} \int \frac{1}{x-a} - \frac{1}{x+a} \, dx$

$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C$

(2) $a \neq 0$ $$\int \frac{dx}{a^2-x^2} = -\int \frac{dx}{x^2-a^2} = \frac{-1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C$$

(3) $$\int \frac{dx}{\sqrt{x^2+a^2}} \underset{\substack{x = a\tan\theta \\ dx = a\sec^2\theta\, d\theta \\ \sqrt{x^2+a^2} = a\sec\theta}}{=} \int \frac{a\sec^2\theta\, d\theta}{a\sec\theta} = \int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta) + C$$
$$= \ln\left(\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right) + C$$

(4) $$\int \frac{dx}{\sqrt{x^2-a^2}} \underset{\substack{x = a\sec\theta \\ dx = a\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-a^2} = a\tan\theta}}{=} \int \frac{a\sec\theta\tan\theta\, d\theta}{a\tan\theta} = \int \sec\theta\, d\theta = \ln\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) + C$$

(5) $$\int \frac{x^2}{1+x^2}\,dx \underset{\text{Long division}}{=} \int \frac{1+x^2-1}{1+x^2}\,dx = \int 1 - \frac{1}{x^2+1}\,dx = x - \arctan x + C$$

(6) $$\int \frac{e^{2x}}{e^x-1}\,dx \underset{\substack{u = e^x \\ du = e^x dx}}{=} \int \frac{u^2}{u-1}\frac{du}{u} = \int \frac{u}{u-1}\,du = \int \left(1 + \frac{1}{u-1}\right) du$$
$$= u + \ln|u-1| + C = e^x + \ln|e^x-1| + C$$

(7) $$\int \frac{dx}{x(\ln x)^2} \underset{\substack{u = \ln x \\ du = \frac{dx}{x}}}{=} \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

(8) $$\int \frac{\sqrt{1+x}}{\sqrt{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}$$
$$= \arcsin x + \int \frac{x\,dx}{\sqrt{1-x^2}} \underset{\substack{u = 1-x^2 \\ du = -2x\,dx}}{=} \arcsin x + \frac{1}{2}\int \frac{-du}{u^{1/2}}$$
$$= \arcsin x - \frac{2}{2}u^{1/2} + C = \arcsin x - \sqrt{1-x^2} + C$$

(9) $$\int \frac{dx}{1+\cos x} = \int \frac{1-\cos x}{1-\cos^2 x}\,dx = \int \frac{1-\cos x}{\sin^2 x}\,dx = \int \frac{dx}{\sin^2 x} - \int \frac{\cos x}{\sin^2 x}\,dx$$
$$= \int \underbrace{\csc^2 x\, dx}_{d(-\cot x)} - \int \frac{\cos x}{\sin^2 x}\,dx \underset{\substack{u = \sin x \\ du = \cos x\, dx}}{=} -\cot x - \int \frac{du}{u^2} = -\cot x + \frac{1}{\sin x} + C$$

(10) $$\int e^{\sqrt{x}}\,dx \underset{\substack{u = \sqrt{x} \\ du = \frac{1}{2}\frac{dx}{\sqrt{x}} = \frac{dx}{2u}}}{=} \int e^u 2u\,du = 2\int u \underbrace{e^u du}_{= dv} = 2\left(ue^u - \int 1 e^u du\right)$$
$$= 2(ue^u - e^u) + C = 2e^{\sqrt{x}}(\sqrt{x}-1) + C$$

(11) $$\int \frac{dx}{(x^2+a^2)^n}$$ for $a > 0$ & $n \geq 2$ integer

Method 1: Use Partial fractions after integration by parts to reduce the multiplicity

$$\int \frac{dx}{(x^2+a^2)^n} = \int \frac{1}{2x} \underbrace{\frac{2x}{(x^2+a^2)^n}dx}_{=dv} = \frac{1}{2x}\frac{1}{(-n+1)(x^2+a^2)^{n-1}} - \int \frac{-1}{2x^2}\frac{1}{(-n+1)(x^2+a^2)^{n-1}}$$

$v = \frac{1}{-n+1}\frac{1}{(x^2+a^2)^{n-1}}$

Write $\frac{1}{x^2(x^2+a^2)^{n-1}} = \frac{A_0}{x} + \frac{A_1}{x^2} + \sum_{k=1}^{n-1} \frac{A_{1,k}x+B_{1,k}}{(x^2+a^2)^k}$ &

repeat the integration by parts whenever $A_{1,k}=0$. & $k \geq 2$.

Method 2: Use a Trigonometric substitution: for x^2+a^2 we use $x = a\tan\theta$

$x^2+a^2 = a^2\sec^2\theta$

$dx = a\sec^2\theta\, d\theta$

$$\int \frac{1}{(x^2+a^2)^n}dx = \int \frac{a\sec^2\theta\, d\theta}{(a^2\sec^2\theta)^n} = \int a^{1-2n}\frac{d\theta}{\sec^{2(n-1)}\theta}$$

$$= \frac{1}{a^{2n-1}}\int \cos^{2(n-1)}\theta\, d\theta$$

Call $I_p := \int \cos^p\theta\, d\theta = \int \cos^{p-1}\theta \underbrace{\cos\theta\, d\theta}_{d(\sin\theta)}$

$$= \cos^{p-1}\theta\sin\theta - \int (p-1)\cos^{p-2}\theta(-\sin\theta)\sin\theta\, d\theta$$

parts

$$= \cos^{p-1}\theta\sin\theta + \int (p-1)\cos^{p-2}\theta\underbrace{\sin^2\theta}_{=1-\cos^2\theta}d\theta = \cos^{p-1}\theta\sin\theta + (p-1)(I_{p-2}-I_p)$$

$$\rightsquigarrow \boxed{I_p = \frac{1}{p}\cos^{p-1}\theta\sin\theta + \frac{p-1}{p}I_{p-2}}$$ Recursion

Our base case is $p=2$ ($2(n-1)$ is even)

Base Case: $p=2$ $\quad I_2 = \int \cos^2\theta\, d\theta \underset{\text{½ angle formula}}{=} \int \frac{1+\cos 2\theta}{2}d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$