Lecture XLI: § 12.2 L'Hôpital Rule. The indeterminacy 0/0

§1. Introduction:

Limit Law: If f & g are two functions defined around $x=a$ with

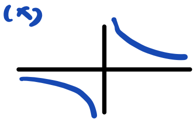
$\lim_{x\to a} f(x) = M$, $\lim_{x\to a} g(x) = N$ and $N \neq 0$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{M}{N}$$

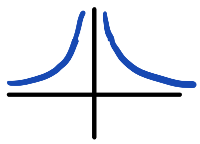
⚠ We can't use this statement if $N=0$.

(1) If $M\neq 0$ & $N=0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ can be $\pm\infty$ or may not exist

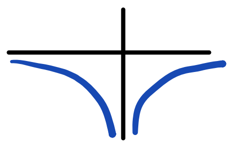
EXAMPLES • $\lim_{x\to 0} \frac{1}{x}$ does not exist



• $\lim_{x\to 0} \frac{1}{x^2} = +\infty$



• $\lim_{x\to 0} \frac{-1}{x^2} = -\infty$



(2) If $M=0$ & $N=0$, anything can happen! (Indeterminacy!)

EXAMPLES: $\lim_{x\to 0} \frac{3x}{x} = 3$; $\lim_{x\to 0} \frac{x^2}{x} = 0$; $\lim_{x\to 0} \frac{x^2}{x^4} = \infty$, $\lim_{x\to 0} \frac{-x^2}{x^4} = -\infty$

$\lim_{x\to 0} \frac{x^2}{x^3}$ does not exist ; $\lim_{x\to 0} \frac{x}{x^2}$ does not exist , $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

• Rational Functions: $\frac{P(x)}{Q(x)}$ with $P(0) = Q(0) = 0$.

Factor P & Q : $P(x) = x^n \tilde{P}(x)$ with $\tilde{P}(0) \neq 0$ $n\geq 0$

$Q(x) = x^m \tilde{Q}(x)$ —— $\tilde{Q}(0)\neq 0$ $m \geq 0$

Then $\lim_{x\to 0} \frac{P(x)}{Q(x)} = \lim_{x\to 0} x^{n-m} \frac{\tilde{P}(x)}{\tilde{Q}(x)} = \begin{cases} 0 & n>m \\ \frac{\tilde{P}(x)}{\tilde{Q}(x)} & n=m \\ \text{does not exist} & \text{if } m>n \text{ \& } m-n=\text{odd} \\ \infty & \text{if } m>n \text{ \& } m-n=\text{even} \end{cases}$

Example $\tilde{P}=\tilde{Q}=1$

- $\boxed{\begin{matrix} m=2 \\ n=1 \end{matrix}}$ $\lim_{x\to 0} \frac{P(x)}{Q(x)} = \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x}$ does not exist $\left(\begin{matrix} \lim_{x\to 0^+} \frac{1}{x} = +\infty \\ \lim_{x\to 0^-} \frac{1}{x} = -\infty \end{matrix} \right)$
- $\boxed{\begin{matrix} m=3 \\ n=1 \end{matrix}}$ $\lim_{x\to 0} \frac{P(x)}{Q(x)} = \lim_{x\to 0} \frac{x}{x^3} = \lim_{x\to 0} \frac{1}{x^2} = +\infty$.

Conclusion: For rational functions, we can bypass $0/0$ indeterminacy with algebraic manipulations!

Example $\lim_{x\to 2} \frac{3x^2-7x+2}{x^2+5x-14} = \lim_{x\to 2} \frac{(x-2)(3x-1)}{(x-2)(x+7)} = \lim_{x\to 2} \frac{3x-1}{x+7} = \frac{5}{9}$

- In other settings: geometry can help:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \overset{\text{“}\sin 0\text{”}}{0}}{x} = (\sin x)'_{(0)} = \cos(0) = 1.$$

Proposition: If f is differentiable at $x=a$, then

$$\underbrace{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}_{0/0 \text{ indet.}} \underset{\Delta x = x-a}{=} \lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x} = f'(a)$$

§2. L'Hôpital's Rule:

L'Hôpital Theorem: Fix a in $\mathbb{R}$ and two functions f, g differentiable in some open interval containing a $(a-\varepsilon, a+\varepsilon)$. Assume that $g'(x) \neq 0$ on this interval except perhaps at $x=a$. If $f(a)=g(a)=0$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$ provided the (RHS) limit exists.

Moreover, if $g'(a)\neq 0$, then (RHS) $= \frac{f'(a)}{g'(a)}$.

Idea: Linear approximation near $x=0$

$$(*) \quad \begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) = f'(a)(x-a) \\ g(x) &\approx g(a) + g'(a)(x-a) = g'(a)(x-a) \end{aligned}$$

These approximations are good if both f' & g' are continuous at $x=a$.

If so, $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)}{g'(a)} \frac{(x-a)}{(x-a)} = \frac{f'(a)}{g'(a)}$

This argument only works if $g'(a)\neq 0$ & the approximations $(*)$ are good.
In general, we will need the Mean Value Theorem (MVT) to prove the Theorem. (next time!)

EXAMPLES ① $\frac{f(x)}{g(x)} = \frac{\operatorname{sen} x}{x}$ is $\frac{0}{0}$ indeterminacy for $x\to 0$.

L'Hôpital gives: $f(x) = \operatorname{sen} x$ $\quad f'(x) = \cos x$
$g(x) = x$ $\quad g'(x) = 1$ (never zero around $x=0$).

So $\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = \boxed{1}$.

② $\frac{f(x)}{g(x)} = \frac{x^2}{x} \xrightarrow[x\to 0]{} x\big|_{x=0} = 0$ (rational function trick!)

Using L'Hôpital: $f'(x) = 2x$, $g'(x) = 1$ (never zero around $x=0$).

So $\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} \frac{2x}{1} = 2\cdot 0 = \boxed{0}$.

③ $\frac{f(x)}{g(x)} = \frac{\tan(3x)}{e^x - 1}$ is $\frac{0}{0}$ indeterminacy for $x\to 0$.

Using L'Hôpital: $f'(x) = \sec^2(3x)\cdot 3 = \frac{3}{\cos(3x)}$
$g'(x) = e^x$ (never zero around $x=0$).

So $\lim_{x\to 0} \frac{\tan(3x)}{e^x-1} = \lim_{x\to 0} \frac{3/\cos(3x)}{e^x} = \frac{3/1}{1} = \boxed{3}$

Observation: We can iterate the process if $\frac{f'(x)}{g'(x)}$ is also an indeterminacy of the form $0/0$ by applying L'Hôpital Rule again. In general:

$$\underbrace{\lim_{x\to a} \frac{f(x)}{g(x)}}_{\frac{0}{0}} = \underbrace{\lim_{x\to a} \frac{f'(x)}{g'(x)}}_{\frac{0}{0}} = \cdots\cdots = \underbrace{\lim_{x\to a} \frac{f^{(k)}(x)}{g^{(k)}(x)}}_{\frac{0}{0}} = \frac{f^{(k+1)}(a)}{g^{(k+1)}(a)} \quad \text{if } g^{(k+1)}(a)\neq 0.$$

Example: ① $\underbrace{\lim_{x\to 0} \frac{1-\cos x}{x^2}}_{\sim \frac{0}{0} \text{ indet}} = \underbrace{\lim_{x\to 0} \frac{\operatorname{sen} x}{2x}}_{\sim \frac{0}{0} \text{ indet}} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{\cos(0)}{2} = \frac{1}{2}$.

$f(x) = 1-\cos x$, $\quad f'(x) = \operatorname{sen} x$, $\quad f''(x) = \cos x$
$g(x) = x^2$, $\quad g'(x) = 2x$, $\quad f''(x) = 2$

(2) $\displaystyle\lim_{x\to 0} \frac{\operatorname{sen} x - x}{x^3} \underset{\text{L'Hôp.}}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \underset{\text{L'Hôp.}}{=} \lim_{x\to 0} \frac{-\operatorname{sen} x}{6x} \underset{\text{L'Hôp.}}{=} \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$

$\frac{0}{0} \qquad \frac{0}{0} \qquad \frac{0}{0}$

⚠ Always verify the $0/0$ indeterminacy!

Example $\displaystyle\lim_{x\to 0} \frac{\operatorname{sen} 4x}{2x+3} = \frac{\operatorname{sen} 0}{3} = 0$ **but** $\displaystyle\lim_{x\to 0} \frac{(\operatorname{sen} 4x)'}{(2x+3)'} = \lim_{x\to 0} \frac{4\cos 4x}{2} = \frac{4}{2} = 2$

Q: What if $x \to \pm\infty$ (i.e. $a = \pm\infty$ in L'Hôpital)

A: Do the change of variables $x = \frac{1}{t}$. Then $x \to \infty$ means $t \to 0^+$
$x \to -\infty$ — $t \to 0^-$

Notice $\displaystyle\lim_{x\to +\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0^+} \frac{f(\frac{1}{t})}{g(\frac{1}{t})}$ & $\displaystyle\lim_{x\to -\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0^-} \frac{f(\frac{1}{t})}{g(\frac{1}{t})}$

The Chain Rule gives:

$$\left(f\left(\tfrac{1}{t}\right)\right)' = f'\left(\tfrac{1}{t}\right)\cdot\left(\tfrac{-1}{t^2}\right) \quad \& \quad \left(g\left(\tfrac{1}{t}\right)\right)' = g'\left(\tfrac{1}{t}\right)\cdot\left(\tfrac{-1}{t^2}\right).$$

So $$\frac{\left(f(\frac{1}{t})\right)'}{\left(g(\frac{1}{t})\right)'} = \frac{f'(\frac{1}{t})\,(-\frac{1}{t^2})}{g'(\frac{1}{t})\,(-\frac{1}{t^2})} = \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \frac{f'(x)}{g'(x)} \quad \text{for } x = \tfrac{1}{t}.$$

This ensures that the rule is valid for $a = \pm\infty$ as well!

Proposition (L'Hôpital Rule for $a = \pm\infty$)

"x large enough"

Assume f & g are differentiable near a (for $x \gg 0$ if $a = +\infty$; for $x \ll 0$ if $a = -\infty$)

"−x large enough"

Assume $g'(x) \neq 0$ in a neighborhood of a (for $x \gg 0$ if $a = +\infty$; for $x \ll 0$ if $a = -\infty$)

If $\displaystyle\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then: $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Example: $\displaystyle\lim_{x\to\infty} \frac{\operatorname{sen}\left(\frac{1}{x}\right)}{\frac{1}{x}} \underset{\text{L'Hôp.}}{=} \lim_{x\to\infty} \frac{\cos\frac{1}{x}\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty} \cos\frac{1}{x} = \cos 0 = 1.$

$\frac{0}{0}$