Lecture XLII: §12.1 The Mean Value Theorem revisited
§12.3 L'Hôpital Rule: Other indeterminate forms

§1. L'Hôpital Rule for $\frac{0}{0}$ indeterminacies:

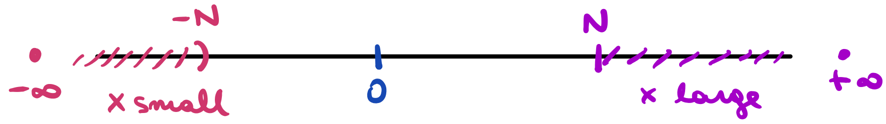
L'Hôpital Theorem: Fix a in $\mathbb{R}$ and two functions f, g differentiable in some open interval $(a-\varepsilon, a+\varepsilon)$ containing a. Assume that $g'(x) \neq 0$ on this interval except perhaps at $x=a$. If $f(a) = g(a) = 0$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{provided the (RHS) limit exists.}$$

Moreover, if $g'(a) \neq 0$, then $(RHS) = \dfrac{f'(a)}{g'(a)}$.

Obs: (Last Time) The rule applies for $a = \pm\infty$ where $\lim_{x\to a} f(x) = \lim_{x \to a} g(x) = 0$.

Neighborhoods



N > 0 large integer.

Q: Why is this rule valid?

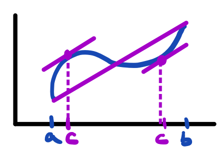
Recall $\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ is a $\frac{0}{0}$ indeterminate & $= \frac{f'(a)}{1}$ if the limit exists.

So we should expect a connection between derivatives & $\frac{0}{0}$ indeterminates. We will use the Mean Value Theorem (MVT) & a generalization of it.

Mean Value Theorem: If $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ & differentiable on (a,b), then there exists $a < c < b$ with $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

slope of tangent line at $(c, f(c))$

slope of the secant line through $(a, f(a))$ & $(b, f(b))$



Special case: $f(a) = f(b)$ ⇝ Rolle's Theorem.

Generalized MVT: Given 2 functions f & g with:

(1) f & g continuous on $[a,b]$

(2) f & g differentiable on (a,b) with $g'(x) \neq 0$ for all x in (a,b)

there exists c in (a,b) with $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

- Usual MVT: Take $g(x) = x$
- Observe: $g(b) \neq g(a)$ since otherwise g would have a critical point in (a,b) & this cannot happen because g' does not vanish on (a,b)

Proof of GMVT: We consider the auxiliary function:

$$F(x) = (f(b)-f(a))(g(x)-g(a)) - (g(b)-g(a))(f(x)-f(a))$$

- F continuous on $[a,b]$ & differentiable on (a,b)
- $F(a) = 0 - 0 = 0$, $F(b) = 0$ $\Big\}$ so $F(a) = F(b)$.

Rolle's Theorem ensures that we can find c in (a,b) with $F'(c)=0$.

But $F'(x) = (f(b)-f(a))\,g'(x) - (g(b)-g(a))\,f'(c)$

So $F'(c)=0$ gives $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$ as we wanted. $\square$

- Next, we use Generalized MVT to confirm L'Hôpital's Rule: We have to analyze 2 cases:

CASE 1 Assume $g'(a) \neq 0$

Then $\dfrac{f(x)}{g(x)} = \boxed{\dfrac{f(x) - \overbrace{f(a)}^{=0}}{g(x) - \underbrace{g(a)}_{=0}}} = \dfrac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \xrightarrow[x\to a]{} \dfrac{f'(a)}{g'(a)}$

$\underset{\text{GMVT}}{=} \dfrac{f'(c)}{g'(c)}$ for some $a<c<x$ (if $x\to a$ then $c \to a$.

here we take not just c but all possible values of x.

Conclusion: $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ (if the (RHS) limit exists!)

CASE 2: By GMVT $\dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$ for some c in (a,x).

So $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{c\to a}\dfrac{f'(c)}{g'(c)}$ if the (RHS) limit exists.

⚠ If the (RHS) limit does not exist, we cannot apply the Rule, meaning we could still have a valid limit for $\dfrac{f(x)}{g(x)}$.

§2. Other Indeterminacies:

GOAL Extend L'Hôpital's Rule from $\frac{0}{0}$ to other indeterminacies.

$0\cdot\infty$, $\infty-\infty$, $\frac{\infty}{\infty}$, 0^0, ∞^0, 1^∞ ($\frac{\infty}{\infty}$ is a special one!)

TYPE 1: $\frac{\infty}{\infty}$ (separate proof!)

TYPE 2 $0\cdot\infty$, 0^0, ∞^0, 1^∞, $\infty-\infty$ (follow from $\frac{0}{0}$ or $\frac{\infty}{\infty}$ rules by algebraic manipulations)

We do type 2 indeterminacies first.

§3. $0\cdot\infty$

IDEA $\underset{f}{0}\cdot\underset{g}{\infty}$ becomes $\frac{\infty}{\infty}$ as $g/(1/f)$, or $\frac{0}{0}$ as $f/(\frac{1}{g})$

Algebra = Divide by reciprocal of one factor

EXAMPLE: $\displaystyle\lim_{x\to0^+} x\ln x = \lim_{x\to0^+}\frac{\ln x}{1/x} \overset{\text{L'Hôp}}{=} \lim_{x\to0^+}\frac{\frac{1}{x}}{-1/x^2} = \lim_{x\to0^+} -x = 0$

$0\cdot-\infty$ $\qquad \frac{\infty}{\infty}$

Other option: $\displaystyle\lim_{x\to0^+}\frac{x}{1/\ln x} = \lim_{x\to0^+}\frac{1}{\frac{-1}{\ln^2 x}\cdot\frac{1}{x}} = \lim_{x\to0^+} -x\ln^2 x$ this is worse!

$0/0$

Conclusion: We should chose a reciprocal with simple derivative!

§4: 3 cases: 0^0, ∞^0, 1^∞

Algebra = Take ln, compute limit & then take exponential of this limit

Why? $\displaystyle\ln\left(\lim_{x\to a}F(x)\right) = \lim_{x\to a}\ln(F(x))$ provided $F(x)>0$ near $x=a$

$\displaystyle\exp\left(\underline{\qquad\qquad}\right) = \lim_{x\to a} e^{F(x)}$.

EXAMPLES 1 $\displaystyle\lim_{x\to0^+} x^x = ?$ 0^0 type

Take ln: $\displaystyle\lim_{x\to0^+}\ln(x^x) = \lim_{x\to0^+} x\ln x \overset{\text{L'Hôp.}}{=} 0$ (earlier example)

$0\quad\infty$

Then $\displaystyle\lim_{x\to0^+} x^x = e^0 = \boxed{1}$

② $\lim_{x\to\infty} x^{1/x} = ?$ $\quad \infty^0$ type

Take ln: $\lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \underbrace{\frac{\ln x}{x}}_{\frac{\infty}{\infty}} \underset{\text{L'Hôp}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

So $\lim_{x\to\infty} x^{1/x} = e^0 = \boxed{1}$

③ $\lim_{x\to 0} (1+ax)^{1/x} = ?$ for $a\neq 0$ $\quad 1^\infty$ type

Take ln: $\lim_{x\to 0} \ln\left((1+ax)^{1/x}\right) = \lim_{x\to 0} \frac{1}{x}\ln(1+ax) = \lim_{x\to 0} \underbrace{\frac{\ln(1+ax)}{x}}_{\frac{0}{0}} =$

$\underset{\text{L'Hôp}}{=} \lim_{x\to 0} \frac{\frac{a}{1+ax}}{1} = a$

So $\lim_{x\to\infty} (1+ax)^{1/x} = \boxed{e^a}$

§5. $\infty - \infty$

Algebra: Take common denominator (if an expression goes to ∞, we can see it as a ratio of 2 functions with denominator going to 0).

EXAMPLE $\lim_{x\to\frac{\pi}{2}} \underbrace{\sec x}_{\pm\infty} - \underbrace{\tan x}_{\pm\infty} = \lim_{x\to\pi/2} \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \lim_{x\to\pi/2} \underbrace{\frac{1-\sin x}{\cos x}}_{0/0}$

$\underset{\text{L'Hôp}}{=} \lim_{x\to\pi/2} \frac{-\cos x}{-\sin x} = \frac{-0}{-1} = \boxed{0}$

§6. $\frac{\infty}{\infty}$

L'Hôpital Rule for $\frac{\infty}{\infty}$: Pick f & g continuous & differentiable functions around $x=a$ with $g'(x)\neq 0$ for all $x\neq a$ near a. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$ provided the (RHS) limit exists.

This rule applied to a in $\mathbb{R}$ & also if $a = \pm\infty$.

Proof: Next Time.