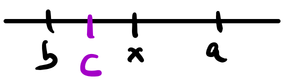
Lecture XLIII: §12.3 L'Hôpital's Rule for $\frac{\infty}{\infty}$
§12.4 Improper integrals

§1 ∞/∞ indeterminacy:

L'Hôpital Rule for $\frac{\infty}{\infty}$: Pick f & g continuous & differentiable functions around $x=a$ with $g'(x) \neq 0$ for all $x \neq a$ near a. If $\lim_{x\to a} f(x) = \lim_{x \to a} g(x) = \infty$, then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{provided the (RHS) limit exists}$$
This rule applied to a in $\mathbb{R}$ & also if $a = \pm\infty$.

Why? First we assume a in $\mathbb{R}$. Say $L = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Assume $x<a$ & pick $b<x$



By the GMVT (Lecture 42), we can find c in (b,x) with
$$\frac{f(x)-f(b)}{g(x)-g(b)} = \frac{f'(c)}{g'(c)}$$

Note: as $b \to a$ so do x & c

Write
$$\frac{f(x)-f(b)}{g(x)-g(b)} = \frac{f(x)}{g(x)} \boxed{\left(\frac{1-\frac{f(b)}{f(x)}}{1-\frac{g(b)}{g(x)}}\right)}$$

- If b is fixed near a, then $\frac{f(b)}{f(x)} \xrightarrow[x\to a]{} \frac{f(b)}{\infty} = 0$ & $\frac{g(b)}{g(x)} \xrightarrow[x \to a]{} \frac{g(b)}{\infty} = 0$

  This implies $\boxed{} \xrightarrow[x\to a]{} \frac{1-0}{1-0} = 1$.

- If b & x are close enough to a, then $\frac{f'(c)}{g'(c)}$ can be made as close to L as we want.

Conclusion
$$\lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{f(x)}{g(x)} \left(\frac{1-\frac{f(b)}{f(x)}}{1-\frac{g(b)}{g(x)}}\right) = \lim_{x\to a^-} \frac{f(x)-f(b)}{g(x)-g(b)}$$
$$= \lim_{x\to a^-} \frac{f'(c)}{g'(c)} = L$$

- Same idea works for the other side limit & also for $a = \pm\infty$.

EXAMPLES: ① $\lim_{x\to\infty} \frac{\ln x}{x^2} \underset{\text{L'Hôp.}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$ $\quad \frac{\infty}{\infty}$

In general, for $p>0$ $\quad \lim_{x\to\infty} \frac{\ln x}{x^p} \underset{\text{L'Hôp}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{p x^{p-1}} = \lim_{x\to\infty} \frac{1}{p x^p} = 0$ $\quad \frac{\infty}{\infty}$

② $n>0$ $\quad \lim_{x\to\infty} \frac{x^n}{e^x} \underset{\text{L'Hôp}}{=} \lim_{x\to\infty} \frac{n x^{n-1}}{e^x} = \dots = \lim_{x\to\infty} \frac{n!\,x}{e^x} = \lim_{x\to\infty} \frac{n!}{e^x} = 0$

$\frac{\infty}{\infty}$ $\quad \frac{\infty}{\infty}$ if $n>1$ $\quad \frac{\infty}{\infty}$

③ $\lim_{x\to 0^+} \frac{-\ln x}{e^{1/x}} \underset{\text{L'Hôp}}{=} \lim_{x\to 0^+} \frac{-1/x}{e^{1/x}\left(\frac{-1}{x^2}\right)} = \lim_{x\to 0^+} \frac{x}{e^{1/x}} = 0.$

$\frac{\infty}{\infty}$ $\quad \frac{0}{\infty}$

④ $\lim_{x\to\infty} \frac{x^2-2}{x+1} = \lim_{x\to\infty} \frac{2x}{1} = \infty.$

$\frac{\infty}{\infty}$ $\quad \frac{\infty}{1}$

§2 Improper integrals

GOAL: Compute $\int_a^b f(t)\,dt$ when the function has issues at a or b, i.e.:

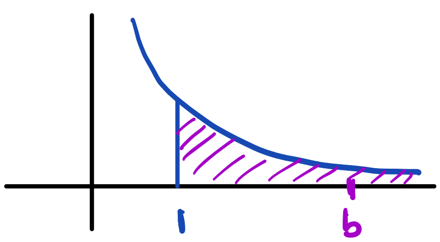
(1) $a=-\infty$ and/or $b=+\infty$

(2) $\lim_{t\to a^+} f(t) = -\infty$ and/or $\lim_{t\to b^-} f(x) = +\infty$, meaning

f has vertical asymptotes at $x=a$ and/or $x=b$.

§2.1 Type (1):

EXAMPLES: ① Calculate the area under the curve $y=\frac{1}{x}$ with $x\geq 1$



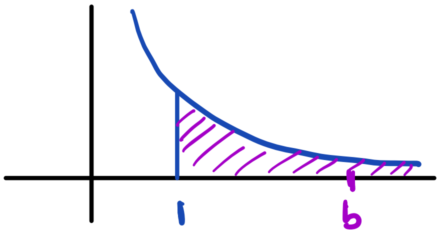
This means $\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

But $\int_1^b \frac{1}{x}\,dx = \ln|x|\Big|_1^b = \ln b - \ln 1 = \ln b$

So Area $= \lim_{b\to\infty} \ln b = \infty$

We write the area as $\int_1^\infty \frac{dx}{x}$ & say this improper integral diverges.

② Calculate the area under the curve $y = \frac{1}{x^2}$ with $x \geq 1$



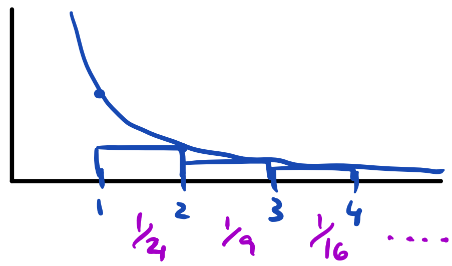
Once again, we need to compute $\lim_{b \to \infty} \int_1^b \frac{dx}{x^2}$

But $\int_1^b \frac{dx}{x^2} = \frac{-1}{x}\Big|_1^b = -\frac{1}{b} - (-1) = 1 - \frac{1}{b}$

So Area $= \lim_{b \to \infty} 1 - \frac{1}{b} = \boxed{1}$

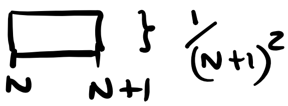
So Area $= \int_1^\infty \frac{dx}{x^2} = 1$ & we say this improper integral converges

Note: Using Riemann Sums (Lower ones)



Areas $= \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$

In general:



Area $= \frac{1}{(N+1)^2}$

This says Area under the curve $\geq$ Sum of the areas of the rectangles

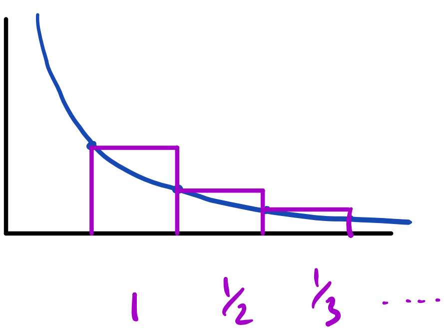
$$1 \geq \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

$$= \lim_{N \to \infty} \sum_{n=2}^{N} \frac{1}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n^2}$$

Ch13 (series)

We will use improper integrals to study convergence of series. (Ch. 13)

In example ① working with Upper Riemann Sums, we see



$$\text{Area} \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

∞

So the series diverges.

3 In general: area under the curve $y=\frac{1}{x^p}$ $p\geq 0$ $p\neq 1$.

$$\int_1^b \frac{dx}{x^p} = \int_1^b x^{-p}\,dx \underset{p\neq 1}{=} \frac{x^{1-p}}{1-p}\Big|_1^b = \frac{1}{1-p}\left(b^{1-p}-1\right)$$

$$\text{Area} = \int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p} = \lim_{b\to\infty}\frac{1}{1-p}\left(b^{1-p}-1\right) = \begin{cases}\infty & \text{if } p<1\\ \frac{-1}{1-p}=\frac{1}{p-1} & p>1\end{cases}$$

Why? $b^{1-p}\xrightarrow[b\to\infty]{}\infty$ if $p<1$ but $b^{1-p}=\frac{1}{b^{p-1}}\xrightarrow[b\to\infty]{}0$ if $p>1$.

Definition $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$.

Similarly: $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$.

Applications (1) Test convergence/divergence of series (Chapter 13)

(2) Laplace Transform: $L_f(p) = \int_0^\infty e^{-px} f(x)\,dx$ of a function f
(used to solve differential eqn)

(3) GAMMA function $\Gamma(p) = \int_0^\infty e^{-x}x^{p-1}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}x^{p-1}\,dx$, $p\neq 0$

But $\int_0^b \underset{u}{e^{-x}}\underbrace{x^{p-1}\,dx}_{dv} \underset{\text{parts}}{=} e^{-x}\frac{x^p}{p}\Big|_0^b - \int_0^b -e^{-x}\frac{x^p}{p}\,dv = e^{-b}\frac{b^p}{p} - 0 + \frac{1}{p}\int_0^b e^{-x}x^p\,dx$

$v=\frac{x^p}{p}$

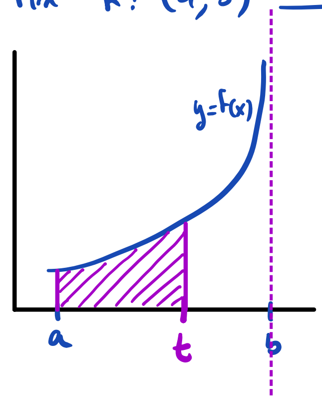
Taking limit as $b\to\infty$ we get $\Gamma(p) = 0-0+\frac{1}{p}\int_0^\infty e^{-x}x^p\,dx = \frac{1}{p}\Gamma(p+1)$

Conclusion: $\boxed{\Gamma(p+1) = p\,\Gamma(p) \quad \text{for any } p\neq 0}$

This function interpolates $(p-1)!$ for $p\geq 1$. because $\Gamma(1) = \int_0^\infty e^{-x}\,dx =$
$= -e^{-x}\Big|_0^\infty = -(0-1) = 1 = 1!$
Thus, $\Gamma(x)$ is a continuous version of the factorial function!

§2.2 Type (II):

Fix $f:(a,b) \longrightarrow \mathbb{R}$ & assume f has a vertical asymptote at $x=b$.



Area under the curve $= \int_a^b f(x)\,dx$ (improper integral)

$$= \lim_{t\to b^-} \int_a^t f(x)\,dx$$

Note: This limit may or may not exist!

Def • If the limit exists, we say the improper integral <u>converges</u>.

• ______ does not exist or it is $\pm\infty$, we say the integral <u>diverges</u>.

EXAMPLES: ① $\int_0^1 \frac{dx}{\sqrt{1-x}}$ $\lim_{x\to 1^-} \frac{1}{\sqrt{1-x}} = +\infty$ & $f(x) = \frac{1}{\sqrt{1-x}} \geq 0$ for $(0,1)$.

$\rightsquigarrow$ asymptote at $x=1$

$$\int_0^t \frac{dx}{\sqrt{1-x}} = -2\sqrt{1-x}\,\Big|_0^t = -2\left(\sqrt{1-t}-\sqrt{1}\right) = 2-2\sqrt{1-t}$$

Then $\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x}} = \lim_{t\to 1^-}\left(2-2\sqrt{1-t}\right) = 2-2\sqrt{1-1} = \boxed{2}$

it converges!

② $\int_0^1 \frac{dx}{1-x}$ $\lim_{x\to 1^-} \frac{1}{1-x} = +\infty$ & $f(x) = \frac{1}{1-x} \geq 0$ on $(0,1)$

$$\int_0^t \frac{dx}{1-x} = -\ln|1-x|\,\Big|_0^t = -\left(\ln|1-t| - \ln|1-0|\right) = -\left(\ln(1-t) - 0\right)$$

Then $\int_0^1 \frac{dx}{1-x} = \lim_{t\to 1^-} \int_0^t \frac{dx}{1-x} = \lim_{t\to 1^-} -\ln(1-t) = -(-\infty) = \boxed{\infty}$

it diverges!

<u>In general</u>: we use the same ideas to define improper integrals when $x=a$ is a vertical asymptote:

Def: $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$ if $x=a$ is a vertical asymptote, but $x=b$ is not.

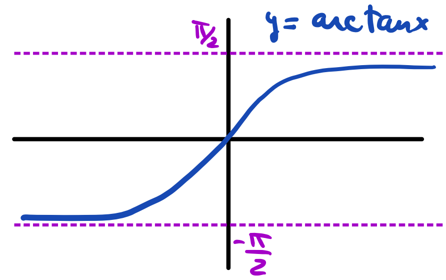
Q: What if we have 2 vertical asymptotes or if $a=-\infty$, $b=+\infty$.

A: Use an intermediate point & additivity of integrals.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \text{for } \underline{\text{ANY}}\ a<c<b$$

⚠ The (RHS) value should <u>not</u> depend on c if the improper integral converges.
This is a subtle point since 1 or both terms on (RHS) are improper integrals.

<u>Typical choice</u>: $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx. \qquad c=0$



<u>EXAMPLE</u>: $\displaystyle f(x) = \frac{1}{1+x^2} \geq 0.$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

$$\int_a^b \frac{dx}{1+x^2} = \arctan x\Big|_a^b = \arctan b - \arctan(a).$$

$$\int_{-\infty}^{c} f(x)\,dx = \lim_{a\to-\infty} \int_a^c \frac{dx}{1-x^2} = \lim_{a\to-\infty} (\arctan c - \arctan(a))$$
$$= \arctan c - \left(-\frac{\pi}{2}\right) = \arctan c + \pi/2$$

$$\int_c^{\infty} f(x)\,dx = \lim_{b\to+\infty} \int_c^b \frac{dx}{1-x^2} = \lim_{b\to\infty} (\arctan b - \arctan c) = \frac{\pi}{2} - \arctan c$$

So $\text{(RHS)} = \left(\arctan c + \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - \arctan c\right) = \boxed{\pi}$ (<u>independent</u> of c)