Lecture XLIV: §13.1 What is an infinite series?

Last Time: we saw examples of infinite series from Riemann Sums of improper integrals.

§1. Basic Definition & examples

Definition: An infinite series (of constants), or series for short, is an expression of the form:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n \qquad (*)$$

- a_n is called the n^{th} term of the series, and it's usually given by a simple formula

Example $a_n = \frac{1}{2^n}$ $n \geq 0$, so series is $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}$ (first index is 0 here)

GOAL: Interpret series such as $(*)$ both formally & exactly, meaning, try to compute its value or determine if its value is ∞ or if it does not exist.

EXAMPLE 1: Infinite decimal expansions

- Finite expansions:

$$0.a_1 a_2 a_3 a_4 \cdots a_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots + \frac{a_n}{10^n}$$

Here: $a_n = 0, 1, 2, \ldots,$ or 9 for each $n \geq 1$.

- Some real numbers in $(0,1)$ have decimal expansions with infinitely many terms

Examples ① $\frac{1}{3} = 0.333\ldots$ means $\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots = \sum_{n=1}^{\infty} \frac{3}{10^n}$

$= 3 \sum_{n=1}^{\infty} \frac{1}{10^n}$

we can manipulate series

So $\frac{1}{9} = \sum_{n=1}^{\infty} \frac{1}{10^n} = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

② $\pi = 3.1415\ldots \rightsquigarrow \pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{1}{5^4} + \cdots = 3 + \sum_{n=1}^{\infty} \frac{a_n}{10^n}$.

EXAMPLE 2: Geometric Series

Q: Can we write $\frac{1}{1-x}$ as a series in x^n?

A YES, via long division

Alternative: Check $(1-x)(1 + x + \cdots + x^{n-1}) = 1 + x + \cdots + x^{n-1} - x - \cdots - x^n$
$= 1 - x^n$

So $\dfrac{1-x^n}{1-x} = 1+x+\dots+x^{n-1} \rightsquigarrow \dfrac{1}{1-x} = 1+\dots+x^{n-1} + \dfrac{x^n}{1-x}$

If $x^n \xrightarrow[n\to\infty]{} 0$ for each fix x (which happens if $|x|<1$)

then: $\dfrac{1}{1-x} = 1+x+\dots+x^{n-1}+\dots = \boxed{\sum_{n=0}^{\infty} x^n}$ Power series expansion of $\frac{1}{1-x}$ for $|x|<1$.

Check: If $x = \frac{1}{10}$, we get $\dfrac{1}{1-\frac{1}{10}} = \dfrac{1}{9/10} = \dfrac{10}{9} \overset{?}{=} 1+\left(\frac{1}{10}+\dots+\frac{1}{10^n}+\dots\right)$

This agrees with $\frac{1}{9} = \frac{1}{10}+\frac{1}{10^2}+\dots = \sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n$ (decimal expansion of $\frac{1}{9}$)

- Variants: replace x by $(-x)$, or x^2.
  - $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} \underset{|-x|=|x|<1}{=} \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n = 1-x+x^2-x^3+x^4-\dots$
  - $\dfrac{1}{1-x^2} \underset{|x^2|=|x|^2<1}{=} \sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty} x^{2n} = 1+x^2+x^4+x^6+\dots$
  - $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} \underset{|-x^2|=|x|^2<1}{=} \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1-x^2+x^4-x^6+\dots$

Q1: Can we manipulate these power series as if they were infinite polynomials?
In particular, can we integrate / differential term-by-term?
If so, we could get a lot of new identities!

EXAMPLES:

① $\ln(1+x) = \displaystyle\int \frac{dx}{1+x} \underset{|x|<1}{=} \int \sum_{n=0}^{\infty}(-1)^n x^n\,dx \underset{\text{SWAP (needs justification!)}}{=} \sum_{n=0}^{\infty}\int(-1)^n x^n\,dx$

$= \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} \underset{\substack{\text{relabel } m=n+1\\ \text{so } m\geq 1}}{=} \sum_{m=1}^{\infty}(-1)^{m-1}\frac{x^m}{m} = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots$

$\rightsquigarrow$ We get a power series expression for $\ln(1+x)$, provided $|x|<1$ & that we can indeed swap $\sum_{n=0}^{\infty}$ & $\int$.

② $\arctan x = \int \frac{dx}{1+x^2} \underset{|x|<1}{=} \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx \underset{\text{SWAP}}{=} \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\, dx$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

⇝ We get a power series expression for $\arctan x$ provided $|x|<1$ & that we can swap $\sum_{n=0}^{\infty}$ & $\int$.

Q2 Once these identities are established, can we evaluate at some x? After all, these series are functions of x!

- Set $x=1$ in ① to guess $\ln 2$ (⚠ we needed $|x|<1$ for the intermediate steps, so we must proceed with caution)

$$\ln(2) = \ln(1+1) \overset{?}{=} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

We will see that this is true (future lecture!)

- Set $x=1$ in ② to guess $\arctan(1)$ (again, intermediate steps used $|x|<1$, so we must be careful)

$$\frac{\pi}{4} = \arctan(1) \overset{?}{=} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

We will see that this is a valid identity (and can be used to compute π up to any desired place)

§2 Application: Differential equations

IDEA Given a differential equation ($y'=y$, $y''+a^2y=0$, etc.) we can guess a solution by (1) proposing $y(x) = \sum_{n=0}^{\infty} a_n x^n$ (Taylor series expansion)

(2) differentiation term-by-term

(3) Writing relations among coefficient to determine them.
⇝ recursion.

Typically, first few terms will be unconstraint.

EXAMPLE 1: $y' = y$

Write $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

$y' = a_1 + 2a_2 x + 3a_3 x + \cdots + (n+1) a_{n+1} x^n + \cdots$

Equate term-by-term: Recursion: $a_{n+1} = \frac{a_n}{n+1}$

$a_0 = a_1 \rightsquigarrow a_1 = a_0$

$a_1 = 2a_2 \rightsquigarrow a_2 = \frac{1}{2} a_1 = \frac{a_0}{2}$ $\rightsquigarrow$ Propose: $a_n = \frac{a_0}{n!}$

$a_2 = 3a_3 \rightsquigarrow a_3 = \frac{a_2}{3} = \frac{a_0}{3 \cdot 2}$

$a_3 = 4a_4 \rightsquigarrow a_4 = \frac{a_3}{4} = \frac{a_0}{4!}$

$\rightsquigarrow y = a_0 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$ $y(0) = a_0$.

But we know $y = \lambda e^x$ is the general solution to $y' = y$. $(\lambda = y(0))$

Conclusion: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (set $0! = 1$)

EXAMPLE 2: $y'' + y = 0$

As before we know the general solution is $y = \alpha \sin(x) + \beta \cos x$
α, β parameters

$y(0) = \beta$ & $y' = \alpha \cos x - \beta \sin x$, so $y'(0) = \alpha$.

Proposed solution: $y = \sum_{n=0}^{\infty} a_n x^n$

$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ $a_0 = y(0)$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1) a_{n+1} x^n + \cdots$ $a_1 = y(1)$

$y'' = 2a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 a_4 x^2 + \cdots + (n+2)(n+1) a_{n+2} x^n + \cdots$

We get: constant term: $a_0 + 2a_2 = 0$

x-term: $a_1 + 3 \cdot 2 a_3 = 0$

x^2-term: $a_2 + 4 \cdot 3 a_4 = 0$

$\vdots$

x^n term: $a_n + (n+2)(n+1) a_{n+2} = 0$. $\rightsquigarrow a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$

We get 2 recursions; one for even indices $(0, 2, 4, \ldots)$ & one for odd indices $(1, 3, 5, \ldots)$

- $a_0 \longrightarrow a_2 = \frac{-a_0}{2} \longrightarrow a_4 = \frac{-a_2}{4.3} = \frac{(-1)^2 a_0}{4!} \longrightarrow a_6 = \frac{-a_4}{6.5} = \frac{(-1)^3 a_0}{6!} \longrightarrow \dots$

We get $\boxed{a_{2k} = \frac{(-1)^k a_0}{(2k)!}}$ for $k \geq 0$

- $a_1 \longrightarrow a_3 = \frac{-a_1}{3!} \longrightarrow a_5 = \frac{-a_3}{5.4} = \frac{(-1)^2 a_1}{5!} \longrightarrow a_7 = \frac{-a_5}{7.6} = \frac{(-1)^3 a_1}{7!} \longrightarrow \dots$

We get $\boxed{a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!}}$ for $k \geq 0$

Conclusion: $y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n \text{ even}} a_n x^n + \sum_{n \text{ odd}} a_n x^n$

REGROUP
(this step requires justification!)

$$= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k a_1}{(2k+1)!} x^{2k+1}$$

$$= a_0 \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \right) + a_1 \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right)$$

Since $a_0 = y(0)$ & $a_1 = y'(0)$, comparing this sum with the general solution $y(x) = a_0 \cos x + a_1 \sin x$ yields:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad \& \quad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$