Lecture XLV: §13.2 Convergent sequences

Last Time: We saw examples of sequences as partial sums of series.

§1 Introduction:

Definition. A sequence is an infinite list $\{a_1, a_2, a_3, \ldots\}$ of real numbers indexed by natural numbers. We write $(a_n)_{n\geq 1}$ (or $(a_n)_{n\geq 0}$ if the first index is 0)

- a_n is called the n^{th} term of the sequence.

EXAMPLES:

		BOUNDED?	INCR/DECR?
1	$a_n = 1$ for all n gives $\{1, 1, \ldots\}$ = constant sequence 1	$A = B = 1$ ✓	const, so incr & decr.
2	$a_n = \frac{1-(-1)^n}{2}$ — $\{1, 0, 1, 0, \ldots\}$	$A=0$, $B=1$ ✓	none
3	$a_n = \frac{n-1}{n}$ — $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$	$A=0$, $B=1$ ✓	str. incr.
4	$a_n = \frac{(-1)^{n+1}}{n}$ — $\{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \ldots\}$	$A=-1$, $B=1$ ✓	none
5	$a_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} \underset{(*)}{=} \frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} = 2\left(1 - \frac{1}{2^{n+1}}\right)$	$A=1$, $B=2$ ✓	str. incr. ($a_{n+1} = a_n + \frac{1}{2^{n+1}}$)
6	$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$	$A=0$, $B=$?? ?	str. incr. ($a_{n+1} = a_n + \frac{1}{n+1}$)
7	$a_n = \left(1 + \frac{1}{n}\right)^n$	$A=0$, $B=$?? ?	??
8	$a_n = n^{th}$ digit in the decimal expansion of π	$A=0$, $B=9$ ✓	??

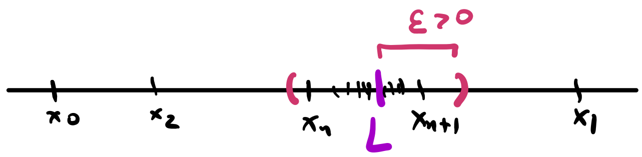
(*) $(1-x)(1+\cdots+x^n) = 1-x^{n+1}$ (last Time) Take $x = \frac{1}{2}$

Definitions:
- A sequence $(a_n)_{n\geq 0}$ is bounded if we can find two constants A & B satisfying $A \leq a_n \leq B$ for all n in $\mathbb{N}$.
- A sequence (a_n) is
  - increasing if $a_n \leq a_{n+1}$ for all n (alt: for all $n \geq n_0$) fixed
  - strictly increasing if $a_n < a_{n+1}$ ——— (——— $n \geq n_0$)
  - decreasing if $a_n \geq a_{n+1}$ ——— (——— $n \geq n_0$)
  - strictly decreasing if $a_n > a_{n+1}$ ——— (——— $n \geq n_0$)

(Alt: we can think of the sequence $(a_n)_{n\geq n_0}$ for this fixed n_0, ie disregard the first few terms of the sequence)

GOAL Understand "long term behavior" of sequences (ie let $n \to \infty$)

Heuristic: $\lim_{n\to\infty} x_n = L$ means as n gets large, the value of x_n gets close to L, meaning $|x_n - L|$ gets close to 0.



$|x_n - L| < \varepsilon$ for $n >> 1$.

Definition: We say $\lim_{n\to\infty} x_n = L$ if for every $\varepsilon > 0$ there exists a positive integer N_0 so that if $n \geq N_0$, then $|x - L| < \varepsilon$.

(Usually: N_0 has a formula in terms of ε.)

Technique to find N_0 = algebraic manipulations, as we did when finding δ for a given ε)

Definition: A sequence $\{x_n\}_{n\geq 0}$ is convergent if it has a limit L in $\mathbb{R}$.
Otherwise, we say it's divergent.

Back to Examples:

① $a_n = 1$ for all n $\rightsquigarrow$ limit $= 1$ $(N_0 = 0)$ $|a_n - 1| = |0| < \varepsilon$ for any n

② $(a_n)_n = 1, 0, 1, 0, \ldots$ so limit cannot exist (the sequence oscillates between 0 & 1)

③ $a_n = \frac{n-1}{n} = 1 - \frac{1}{n} \xrightarrow[n\to\infty]{} 1 - 0 = 1$

Formally: given $\varepsilon > 0$ pick N_0 with $\frac{1}{N_0} < \varepsilon$, so $\frac{1}{\varepsilon} < N_0$.

For example, we can pick $N_0 = 1 + \lceil \frac{1}{\varepsilon} \rceil$ will do ($\lceil\ \rceil$ = ceiling.)

④ $a_n = \frac{(-1)^{n+1}}{n}$ $|a_n| = \frac{1}{n} \to 0$ so $a_n \xrightarrow[n\to\infty]{} 0$ pick $N_0 = 1 + \lceil \frac{1}{\varepsilon} \rceil$

⑤ $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = \underbrace{2\left(1 - \frac{1}{2^{n+1}}\right)}_{= 2 - \frac{1}{2^n}} \xrightarrow[n\to\infty]{} 2(1-0) = 2$

Given $\varepsilon > 0$ want $|a_n - 2| = |\frac{-1}{2^n}| < \varepsilon$ for $n \geq N_0$

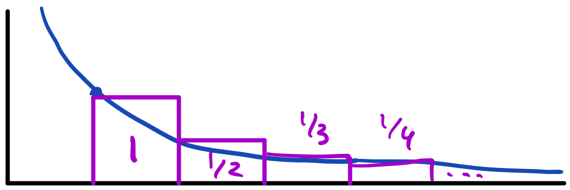
Need $\frac{1}{2^n} < \varepsilon$, so $\frac{1}{\varepsilon} < 2^n$ Take ln $\ln\left(\frac{1}{\varepsilon}\right) < n \ln 2$

This gives $\dfrac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2} < n$ Take $N_0 = 1 + \left\lceil \dfrac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2} \right\rceil$.

(6) $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

Claim (Last lecture) $\lim\limits_{n\to\infty} 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \infty$

Why? Use improper integral & upper Riemann Sums:



$$\text{Area} = \int_1^\infty \frac{1}{x}\,dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$
(Upper R.S.)

(7) $a_n = \left(1 + \frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e$

Why? We saw $\lim\limits_{x\to 0^+} (1+x)^{1/x} = e$ $\quad 1^\infty$

(Take ln: $\lim\limits_{x\to 0^+} \frac{1}{x}\ln(1+x) \underset{\substack{\downarrow \\ \text{l'Hôp.}}}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{1+x}}{1} = 1$ ⤳ Take exponential)

$\frac{0}{0}$

Take $x = 1/n \longrightarrow 0$ means $n \to \infty$.

(8) $a_n = n^{th}$ digit of π is divergent (later).

Idea: Since a_n is discrete, the only way it will converge is if it were constant

But this will make π a rational number, and it is not!

Next Time: Main techniques to determine limits.