Lecture XLVI: §13.2 (cont) Convergent sequences

Recall $\lim_{n\to\infty} x_n = L$ if for every $\varepsilon>0$ we can find N_0 in $\mathbb{Z}_{\geq 0}$ so that if $n \geq N_0$ then $|x_n - L| < \varepsilon$.

Last lecture: we saw many examples, but what are the main techniques?

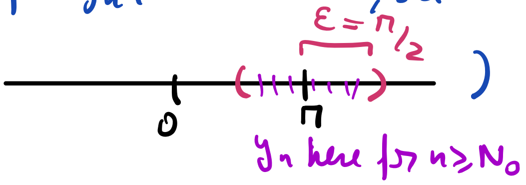
§1. Limit Laws:

Proposition: If $\lim_{n\to\infty} x_n = L$ & $\lim_{n\to\infty} y_n = M$, then the sequences $\{x_n + y_n\}$, $\{x_n - y_n\}$, $\{x_n \cdot y_n\}$ are convergent and

(1) $\lim_{n\to\infty} x_n \pm y_n = L \pm M$; (2) $\lim_{n\to\infty} x_n \cdot y_n = L \cdot M$

Furthermore, if $M \neq 0$, the sequence $\left\{\frac{x_n}{y_n}\right\}_{n \geq n_0}$ converges & $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{L}{M}$

Proof: Variant of ε/δ game for limit laws of functions.

Note: even if $M \neq 0$, the first few terms of $\{y_n\}$ can be 0, but after some point they are all $\neq 0$ (Ex: $M>0$ )

EXAMPLES ① $z_n = \dfrac{n^2+4}{5n^2+6n+7} \underset{\substack{\text{divide by } n^2 \\ \text{numerator \& denominator}}}{=} \dfrac{\overbrace{1+4/n^2}^{=x_n}}{\underbrace{5+\frac{6}{n}+\frac{7}{n^2}}_{=y_n}} \xrightarrow[n\to\infty]{} \dfrac{1}{5}$

② $z_n = \sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n})\dfrac{(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \dfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}} \xrightarrow[n\to\infty]{} 0$

§2. Squeeze Theorem:

Theorem: Suppose 3 sequences $\{a_n\}_n$, $\{b_n\}_n$, $\{x_n\}_n$ satisfy:

(1) $a_n \leq x_n \leq b_n$ for all n large enough

(2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$

Then, $\{x_n\}_{n\in\mathbb{N}}$ is convergent & $\lim_{n\to\infty} x_n = L$

Note: Same happens if in (2) we have $L=\infty$, then $\{x_n\}$ also has limit $=\infty$ & so it's divergent.

EXAMPLES: (1) $x_n = \frac{1}{n!}$ $\quad \underset{\downarrow \atop 0}{a_n = 0} \leq x_n \leq \underset{\downarrow \atop 0}{\frac{1}{n}} = b_n$ so $x_n \xrightarrow[n\to\infty]{} 0$.

(2) $x_n = \frac{a^n}{n!}$ for $a>0$ fixed. **Claim**: $x_n \xrightarrow[n\to\infty]{} 0$

Why? $0 \leq x_n$ so we take $a_n = 0$ in Squeeze Theorem

Need to find b_n with $x_n \leq b_n$ for all n large enough with $\lim\limits_{n\to\infty} b_n = 0$

IDEA: $x_n = \frac{a}{n} \underbrace{\frac{a}{n-1} \cdots \frac{a}{2}\,\frac{a}{1}}_{=x_{n-1}}$

Pick $N_0 > 0$ with $\frac{a}{N_0} < \frac{1}{2}$ & write any $n \geq N_0$ as $n = N_0 + k$ with $k \geq 0$.

So $x_n = \frac{a^n}{n!} = \frac{a^{N_0+k}}{(N_0+k)!} = \frac{a^{N_0}}{N_0!} \underbrace{\boxed{\frac{a}{N_0+1}}}_{<\frac{a}{N_0}<\frac{1}{2}} \cdots\cdots \underbrace{\boxed{\frac{a}{N_0+k}}}_{<\frac{a}{N_0}<\frac{1}{2}} < \frac{a^{N_0}}{N_0!}\left(\frac{1}{2}\right)^k$

Set $b_n = \begin{cases} \frac{a^{N_0}}{N_0!}\left(\frac{1}{2}\right)^{n-N_0} & \text{if } n \geq N_0 \\ 0 & \text{otherwise} \end{cases}$

Then $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \underbrace{\boxed{\frac{a^{N_0}}{(N_0)!}}}_{\text{FIXED}} \underbrace{\frac{1}{2^{n-N_0}}}_{\xrightarrow[n\to\infty]{} 0} = 0$

Since $x_n \leq b_n$ for $n \geq N_0$, by the Squeeze Theorem, $x_n \xrightarrow[n\to\infty]{} 0$.

Note: $\{x_n\}$ grows for a while if a is large, but then it starts decreasing, right about when $\frac{a}{n} < 1$.

§3. Convergence criteria:

Theorem 1: Assume $\{x_n\}$ is increasing ($x_n \leq x_{n+1}$ for all n large enough).
Then, $\{x_n\}$ is convergent if and only if ($\Leftrightarrow$) it is bounded (from above).

Theorem 2 Assume $\{x_n\}$ is decreasing ($x_n \geq x_{n+1}$ for all n large enough).
Then, $\{x_n\}$ is convergent if and only if it is bounded (from below).

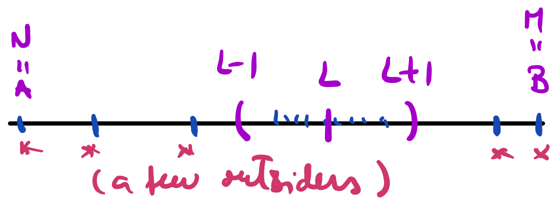
Observe: It suffices to confirm Theorem 1. Taking $y_n = -x_n$ will confirm Theorem 2

- To confirm Theorem 1 we need to prove both implications: The direction ($\Rightarrow$) is true in general, so we write it separately:

Lemma: If a sequence converges, then it is bounded.

Why? Write $\lim_{n\to\infty} x_n = L$ & take $\varepsilon = 1$. We can find N_0 so that
$L-1 < x_n < L+1$ for all $n \geq N_0$.

For $n < N_0$ we need different bounds.



Pick $M = \max\{x_1, x_2, \ldots, x_{N_0-1}\}$
$N = \min\{x_1, x_2, \ldots, x_{N_0-1}\}$

so $N \leq x_n \leq M$ for $n < N_0$

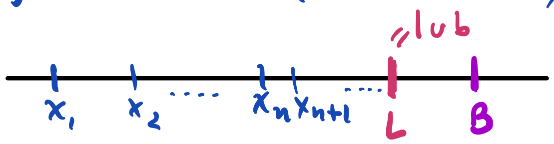
Take $A = \min\{N, L-1\}$
$B = \max\{M, L+1\}$
to conclude $A \leq x_n \leq B$ for all n.

This confirms that the sequence $\{x_n\}$ is bounded

Proof of Theorem 1: By double implication.

($\Rightarrow$) Is the statement of the Lemma.

($\Leftarrow$) Assume $\{x_n\}$ is bounded, say $x_n \leq B$ for all n. We want to find the limit



(after tossing out the first few terms, we can assume $x_n \leq x_{n+1}$ for all n)

- The lower the B = upper bound, the better.

Claim: We can pick the least upper bound $L = \inf\{B : x_n \leq B \text{ for all } n\}$

This L will agree with $\lim_{n\to\infty} x_n$.

We can pick L = l.u.b. by the way we construct $\mathbb{R}$.

Least Upper Bound (l.u.b.) axiom for $\mathbb{R}$: Every non-empty set S of $\mathbb{R}$ that has an upper bound also has a least upper bound = infimum (upper bounds for S)

infimum = Tightest upper bound

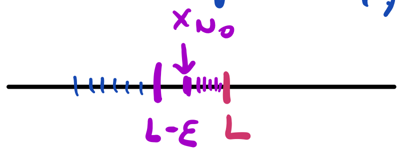
<u>Obs</u>: $\mathbb{Q}$ = rational numbers don't have this property

$S = \{x \text{ in } \mathbb{Q} \text{ with } x^2 < 2\}$ has upper bounds $(2, 3, 4, \ldots)$

But it has no l.u.b because $\sqrt{2}$ is not in $\mathbb{Q}$.

Why is $L = \lim_{n\to\infty} x_n$? Pick $\varepsilon > 0$. Want to find N_0 with $|x_n - L| < \varepsilon$ if $n \geq N_0$.

Definition of lub says $L - \varepsilon$ is not an upper bound for $\{x_n\}$, meaning we can find N_0 with $L - \varepsilon < x_{N_0} \leq L$



But now, $x_{N_0} \leq x_{N_0+1} \leq x_{N_0+1} \cdots$, so $L - \varepsilon < x_n$ for all $n \geq N_0$

Since L is an upper bound $x_n \leq L$ for $n \geq N_0$

<u>Conclusion</u>: $L - \varepsilon < x_n < L$ for $n \geq N_0$ so $|x_n - L| < \varepsilon$ for $n \geq N_0$ as we wanted!

<u>§4 Examples</u>:

① $x_n = \dfrac{\sin(n!)}{3^n} \longrightarrow 0$ By Squeeze Theorem.

$$\underset{\substack{\downarrow n\to\infty \\ 0}}{\frac{-1}{3^n}} \leq x_n \leq \underset{\substack{\downarrow n\to\infty \\ 0}}{\frac{1}{3^n}} \qquad \text{so} \quad x_n \xrightarrow[n\to\infty]{} 0$$

② $x_n = 1 - \frac{1}{3^n} \longrightarrow 1$ We can see this in 2 ways:

(1) Limit Laws: $\lim_{n\to\infty} 1 - \frac{1}{3^n} = \lim_{n\to\infty} 1 - \lim_{n\to 0} \frac{1}{3^n} = 1 - 0 = 1$

(2) $\left.\begin{array}{l} \bullet\ x_n \text{ increasing } \quad 1 - \frac{1}{3^n} \leq 1 - \frac{1}{3^{n+1}} \quad \text{because } 3^n \leq 3^{n+1}. \\ \bullet\ x_n \text{ is bounded above by } 1 \end{array}\right\}$ By Theorem 1 we converge.

In fact $1 = \text{l.u.b of } \{1 - \frac{1}{3^n}\}$. so the limit is 1.

③ $x_n = \dfrac{n^n}{3^{n^2}}$ Write $x_n = \dfrac{n^n}{(3^n)^n} = \left(\dfrac{n}{3^n}\right)^n \sim 0^\infty$

Take $f(x) = \dfrac{x^x}{3^{x^2}} \sim \dfrac{\infty}{\infty}$ as $x \to \infty$ So we can use L'Hôpital!

Take ln: $\ln(f(x)) = x\left(\ln\frac{x}{3^x}\right) = x\left(\ln x - \ln(3^x)\right)$
$= x\ln x - x^2\ln(3)$
$= x^2\left(\frac{\ln x}{x} - \ln 3\right)$

We know $\lim_{x\to\infty}\underbrace{\frac{\ln(x)}{x}}_{\frac{\infty}{\infty}} \underset{\text{L'Hôp.}}{=} \lim_{x\to\infty}\frac{1/x}{1} = 0$

So $\ln(f(x)) = \underbrace{x^2}_{\substack{\downarrow x\to\infty \\ \infty}}\Big(\underbrace{\frac{\ln x}{x}}_{\substack{\downarrow x\to\infty \\ 0}} - \underbrace{\ln 3}_{>1}\Big) \xrightarrow[x\to\infty]{} -\infty$

Then $\lim_{x\to\infty}\frac{x^x}{3^{x^2}} = e^{-\infty} = 0.$

In particular $\frac{n^n}{3^{n^2}} \xrightarrow[n\to\infty]{} 0$

(4) $x_n = \frac{n^n}{3^{n^a}}$ for a fixed

(1) $a<0$ Write $a=-b$ with $b>0$ so $3^{n^a} = 3^{n^{-b}} = 3^{\overset{n\to\infty}{\xrightarrow{} 0}{1/n^b}} \xrightarrow[n\to\infty]{} 1$

Conclude $x_n \longrightarrow \frac{\infty}{1} = \infty.$

(2) $a=0$ Then $n^0=1$ & so $x_n = \frac{n^n}{3} \xrightarrow[n\to\infty]{} \infty$

(3) $a>0$: then $3^{n^a} \sim 3^\infty = \infty$ & $n^n \sim \infty$, so we can use the L'Hôpital approach with $f(x) = \frac{x^x}{3^{x^a}}$.

Take ln: $x\ln x - x^a\ln 3 = x^a\left(x\frac{\ln x}{x^a} - \ln 3\right) = x^a\left(\frac{\ln x}{x^{a-1}} - \ln 3\right)$

- If $a>1$: $\frac{\ln x}{x^{a-1}} \sim \frac{\infty}{\infty}$, so by L'Hôpital we get

$$\lim_{x\to\infty}\frac{\ln x}{x^{a-1}} = \lim_{x\to\infty}\frac{1/x}{(a-1)x^{a-2}} = \lim_{x\to\infty}\frac{1}{\underbrace{(a-1)}_{>0}\underbrace{x^{a-1}}_{\to\infty}} = 0$$

So $\lim_{x\to\infty} x^a\left(\frac{\ln x}{x^{a-1}} - \ln 3\right) \longrightarrow -\infty.$ $\leadsto$ Exponentiate to get $\lim_{x\to\infty}\frac{x^x}{3^{x^a}} = 0.$

- If $a=1$: $x_n = \left(\frac{n}{3}\right)^n \sim \infty^\infty$ so $\lim_{n\to\infty} x_n = \infty.$

- **If $0<a<1$:** $\quad \dfrac{\ln x}{x^{a-1}} = x^{\overset{>0}{1-a}} \ln x \sim \infty \cdot \infty = \infty.$

So $\displaystyle\lim_{x\to\infty} x^a\left(\frac{\ln x}{x^{a-1}} - \ln 3\right) = \infty \cdot (\infty - \ln 3) = \infty.$

Exponentiate to get: $\displaystyle\lim_{x\to\infty} \frac{x^x}{3^{x^a}} = e^{\infty} = \infty.$

Conclusion: $\displaystyle\lim_{n\to\infty} \frac{n^n}{3^{n^a}} = \begin{cases} \infty & \text{if } a \le 1 \\ 0 & \text{if } a > 1 \end{cases}$