Lecture XLVII: §13.3 Convergent & divergent series

§1 From sequences to series:

Definition: If $\{a_n\}_n = \{a_1, a_2, a_3, \ldots\}$ is a sequence, the series with general term a_n is the expression $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

Example (Lecture 45) $\sum_{n=0}^{\infty} \frac{1}{10^n} = 1 + \frac{1}{10} + \frac{1}{10^2} + \cdots$ = geometric series with general term $a_n = \frac{1}{10^n}$

$$= \frac{1}{1 - 1/10} = \frac{10}{9}$$

Q: How to interpret $\sum_{n=1}^{\infty} a_n$?

A Consider the sequence of partial sums

$S_1 = a_1$

$S_2 = a_1 + a_2$

$\vdots$

$S_n = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j$

Example: $S_1 = 1$

$S_2 = 1 + \frac{1}{10}$

$\vdots$

$S_n = 1 + \frac{1}{10} + \cdots + \frac{1}{10^{n-1}}$ (*)

(*) $1 - x^n = (1-x)(1 + x + \cdots + x^{n-1})$ for $x = \frac{1}{10}$ gives $S_n = \frac{1 - (1/10)^n}{1 - 1/10} = \frac{10}{9}\left(1 - \frac{1}{10^n}\right)$

⇝ Use the partial sums to define the series as their limit.

Def: $$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j = \lim_{n\to\infty} S_n.$$

Example above: $S_n = \frac{10}{9}\left(1 - \boxed{\frac{1}{10^n}}\right) \longrightarrow \frac{10}{9}(1-0) = \frac{10}{9}$ (by Limit Laws)

($\frac{1}{10^n} \to 0$ as $n\to\infty$)

So $\sum_{j=0}^{\infty} \frac{1}{10^j} = \frac{10}{9}$

Definition: If the limit of partial sums exists and it's finite ($= L$ in $\mathbb{R}$), we say that the series converges (to L). Alternative: L is the sum of the series.

If the partial sums have no limit or its limit is $\pm\infty$, we say that the the series diverges.

Note: This definition is very similar to our approach to improper integrals.

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx \quad \rightsquigarrow \quad \sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j .$$

We will use the connection to Riemann Sums to compute various sums of series & show divergence (eg harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because $\int_1^\infty \frac{1}{x}dx$ does)

§2 Geometric Series:

Recall: $(1-x)(1+x+\cdots+x^n) = 1-x^{n+1} \underset{x\neq 1}{\rightsquigarrow} 1+x+\cdots+x^n = \frac{1-x^{n+1}}{1-x}$ for $x\neq 1$

⚠ $\lim_{n\to\infty} \frac{1+x^{n+1}}{1-x} = \frac{1}{1-x}\left(1-\lim_{n\to\infty} x^{n+1}\right)$ only gives a number when $|x|<1$

- So Geometric Series $= \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$ whenever $|x|<1$

- Furthermore, the series diverges if $|x|\geq 1$:
  - x^{n+1} oscillates if $x\leq -1$ so it has no limit as $n\to\infty$
  - $\lim_{n\to\infty} x^{n+1} = \infty$ if $x\geq 1$.

Variant: What if we start somewhere other than 0?

$$\sum_{n=k}^{\infty} x^n = x^k + x^{k+1} + \cdots = x^k(1+x+x^2+\cdots)$$
$$= x^k \sum_{j=0}^{\infty} x^j = \boxed{\frac{x^k}{1-x}} \quad \text{for } |x|<1.$$

Note: If $k=0$, then $x^0=1$ & we recover the original formula.

EXAMPLE: $\sum_{n=3}^{\infty} \frac{1}{2^n} = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots = \frac{1}{8}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right) = \frac{1}{8}\,\frac{1}{1-1/2} = \frac{1}{4}$

Alternative approach: Add and substract missing terms:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \cdots = \left(1+\frac{1}{2}+\frac{1}{2^2}\right) + \left(\frac{1}{2^3}+\frac{1}{2^4}+\cdots\right)$$

So $\sum_{n=3}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} - \left(1+\frac{1}{2}+\frac{1}{4}\right) = 2 - \frac{7}{4} = \frac{1}{4}$

Same alternative approach works in general.

$$\sum_{n=0}^{\infty} x^n = (1+x+x^2+\cdots+x^{k-1}) + (x^k + x^{k+1} + \cdots) \quad \text{for } |x|<1$$

$$\text{So } \sum_{n=k}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - (1+x+\cdots+x^{k-1}) \underset{|x|<1}{=} \frac{1}{1-x} - \frac{1-x^k}{1-x} = \frac{x^k}{1-x}$$

§3. Application:

Proposition: The only numbers with decimal expansions with repeated patterns are rational numbers.

Examples: $0.33\cdots = 0.\overline{3} = \frac{1}{3}$ $\quad 1.11\cdots = 1.\overline{1} = \frac{10}{9}$

Why? Say z has a decimal expansion with repeated patterns & $0 \le z < 1$.

Write $z = 0.\underbrace{a_1 a_2 a_3 \cdots a_n}_{\text{non repeated part}} \underbrace{\boxed{a_{n+1} \cdots a_{n+s}}}_{s \text{ terms in repeating pattern}} \underbrace{\boxed{a_{(n+s)+1} \cdots a_{(n+2s)}}}_{\text{repeated pattern}} \cdots$

Write $b_1 \cdots b_s$ for the repeated pattern

$$\text{So } z = \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \underbrace{\sum_{k=1}^{n} \frac{a_k}{10^k}}_{a \text{ in } \mathbb{Q}} + \underbrace{\frac{b_1}{10^{n+1}} + \cdots + \frac{b_s}{10^{n+s}}}_{y = \frac{1}{10^n}\left(\frac{b_1}{10} + \cdots + \frac{b_s}{10^s}\right)} + \underbrace{\frac{b_1}{10^{n+s+1}} + \cdots + \frac{b_s}{10^{n+2s}}}_{\frac{1}{10^s} y} + \cdots$$

$$= a + y + \frac{y}{10^s} + \frac{y}{10^{2s}} + \cdots$$

$$= a + y \underbrace{\left(1 + \frac{1}{10^s} + \frac{1}{10^{2s}} + \cdots\right)}_{\text{geometric series for } x = \frac{1}{10^s}} = a + y \cdot \frac{1}{1 - 1/10^s} \quad \text{in } \mathbb{Q}$$

- Now, if z has a repeated pattern but $z<0$ or $z>1$, we can add an integer N to it ($\lfloor z \rfloor$) so that $z+N$ has a repeated pattern (same as z) and now $0 \le z+N < 1$. If $z+N$ is in $\mathbb{Q}$, so is z.

Q: How to build a decimal expansion for a rational numbers?

Given z we write $z = N + \boxed{\frac{p}{q}}$ with N integer & $0 < q < p$ or $p = 0$.

↳ rational & in $[0,1)$

by using long division on $z = \frac{a}{b}$.

So decimal expansion of z & $\frac{p}{q}$ agree.

By our Proposition, we should have a repeated pattern for $\frac{p}{q}$. How do we find it?

EXAMPLE:

$$\begin{array}{r} 3.\boxed{1}42857 \\ 7 \,\overline{)\, 22} \\ -\,21 \\ \hline \textcircled{1}0 \\ -\,7 \\ \hline 30 \\ -\,28 \\ \hline 20 \\ -\,14 \\ \hline 60 \\ -\,56 \\ \hline 40 \\ -\,35 \\ \hline 50 \\ -\,49 \\ \hline \textcircled{1} \end{array}$$

repeated remainder

$$\frac{22}{7} = 3.142857\,142857\ldots$$

$$= 3.\overline{142857}$$

$$= \underset{a}{3} + \overbrace{\left(\frac{1}{10} + \frac{4}{10^2} + \frac{2}{10^3} + \frac{8}{10^4} + \frac{5}{10^5} + \frac{7}{10^6}\right)}^{=y} \cdot \left(1 + \frac{1}{10^6} + \frac{1}{10^{36}} + \cdots\right)$$

$$= a + y\left(\frac{1}{1 - \frac{1}{10^6}}\right) = a + y\,\frac{10^6}{10^6 - 1}$$

$a = 3$

$y = \frac{142857}{10^6}$

General Procedure: Given z rational

STEP 1: Use long division to write $z = N + x$ with N integer, x in $\mathbb{Q}$, $0 \leq x < 1$

STEP 2: $x = \frac{p}{q}$ $0 < q < p$. & compute the decimal expansion of x by doing long division, adding 0's to the remainder. (eg: $\frac{1}{100} = 0.001$) Since the remainders are $0, 1, 2, \ldots, p-1$, this process cannot go forever without repetitions. The pattern will then consist of all ratios that produced the string of remainders when the repetition first took place.

Consequence: π, $\sqrt{2}$ are not rational, so they decimal expansions have no repeated patterns!