Lecture XLVIII: §13.3 (cont) Convergent & divergent series
§13.4 General Properties

Recall Geometric series: $\sum_{i=0}^{n} x^i = 1+x+x^2+\cdots = \begin{cases} \frac{1}{1-x} & \text{if } |x|<1 \\ \text{diverges} & \text{otherwise} \end{cases}$

Motivation $\frac{1}{1-x}$ is not a polynomial but can be written as a geometric series, so it is a limit of polynomials ($\lim_{n\to\infty} \sum_{j=0}^{n} x^j$)

$\frac{1}{1-x} = 1+x+x^2+\cdots$ means $1 = (1-x)(1+x+x^2+\cdots)$

$= 1+\cancel{x}+\cancel{x^2}+\cancel{x^3}+\cdots -\cancel{x}-\cancel{x^2}-\cancel{x^3}-\cdots$

distribute   all terms but 1 get cancelled

We have a problem as soon as x goes from a symbolic variable to a numerical one precisely because we won't be able to cancel & $1 \neq \infty - \infty$ if $x>1$

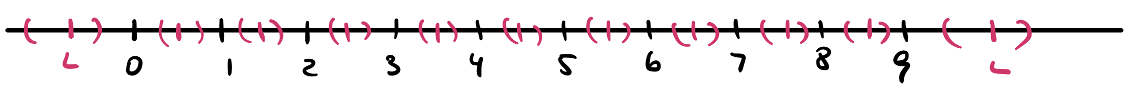
§1 Applications of geometric series

Application 1: Rational numbers = numbers with decimal expansions that have repeated patterns
(Last Time)

Application 2: The sequence $\{a_n\}_n = \{n^{th}$ decimal digit of $\pi\}$ diverges

$a_n = 0, 1, 2, \ldots,$ or 9.

Q: What can the limit be?



CASE 1: L is not in $\{0,1,2,\ldots,9\}$

Take $\varepsilon = \min\{|0-L|, |1-L|, |2-L|, \ldots |9-L|\} > 0$

Then $|0-L| \geq \varepsilon$, $|1-L| \geq \varepsilon$, $\ldots$, $|9-L| \geq \varepsilon$ forces $|a_n - L| \geq \varepsilon$ for all n

So the limit cannot be L.

CASE 2: L is one of $\{1,2,\ldots,9\}$

Take $\varepsilon = 1$. If $\lim_{n\to\infty} a_n = L$ this means we can find $N_0 \geq 1$ with

$|a_n - L| < 1$ for $n \geq N_0$. This forces $a_n = L$ for all $n \geq N_0$ (otherwise $|a_n - L| \geq 1$)

This will say the decimal expansion of π has a repeated pattern $(=L)$ but this cannot happen because π is not rational.

Conclusion: We cannot have a limit, so $\{a_n\}_n$ diverges.

§2. Manipulation of series:

Recall $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{k=1}^{N} a_k$ (limit of sequence of partial sums)

Q: Can we manipulate series as we did with limit laws?

A: For now, we can answer YES in 3 cases:

① Scalar Multiplication:

If λ in $\mathbb{R}$ & $\sum_{n=1}^{\infty} a_n = L$, then $\sum_{n=1}^{\infty} (\lambda a_n) = \lambda \sum_{n=1}^{\infty} a_n = \lambda L$

new series

Why? Works for partial sums + we have limit laws.

② Addition / Substraction:

If $\sum_{n=1}^{\infty} a_n = L$ & $\sum_{n=1}^{\infty} b_n = T$, then $\sum_{n=1}^{\infty} (a_n \pm b_n) = L \pm T$

Why? $\sum_{n=0}^{\infty} (a_n \pm b_n) = \lim_{N\to\infty} \sum_{k=1}^{N} (a_k \pm b_k) = \lim_{N\to\infty} (a_1 \pm b_1) + \cdots + (a_N \pm b_N)$

$$\overset{\text{reorder}}{=} \lim_{N\to\infty} (a_1 + \cdots + a_N) \pm (b_1 + \cdots + b_N)$$

$$= \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = L \pm T.$$

both limits exist so we use Limit Laws

Q: Other examples?

EXAMPLE 1: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = 1$

Why? General Term $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (use the partial fraction decomposition $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$)

So $S_1 = a_1 = 1-\frac{1}{2}$

$S_2 = a_1 + a_2 = \left(1-\cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \tfrac{1}{3}\right) = 1-\frac{1}{3}$

$S_3 = S_2 + a_3 = \left(1-\cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \tfrac{1}{4}\right) = 1-\frac{1}{4}$

In general: $S_n = 1 - \frac{1}{n+1}$

- True for $n=1$
- If true for n, then true for $n+1$

$S_{n+1} = S_n + a_{n+1} = \left(1-\cancel{\tfrac{1}{n+1}}\right) + \left(\cancel{\tfrac{1}{n+1}} - \tfrac{1}{n+2}\right) = 1 - \frac{1}{n+2} = 1 - \frac{1}{(n+1)+1}$

induction argument

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - 0 = \boxed{1}$

This is an example of a Telescopic Series.

General Telescopic Series: $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = f(n) - f(n+1)$ for some function f

Then $S_n = a_1 + a_2 + \cdots + a_n = (f(1) - \cancel{f(2)}) + (\cancel{f(2)} - \cancel{f(3)}) + \cdots + (\cancel{f(n)} - f(n+1))$

$= f(1) - f(n+1)$

Proposition: A Telescopic Series $\displaystyle\sum_{n=1}^{\infty} (f(n) - f(n+1))$ converges if and only if $\displaystyle\lim_{n\to\infty} f(n) = L$ in $\mathbb{R}$

If so $\displaystyle\sum_{n=1}^{\infty} (f(n) - f(n+1)) = f(1) - L$

Example above: $f(x) = \frac{1}{x}$ & $L = 0$, $f(1) = 1$.

§3. General properties:

Proposition If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges then $a_n \xrightarrow[n\to\infty]{} 0$

Why? Write $a_n = S_n - S_{n-1} \xrightarrow[n\to\infty]{} L - L = 0$ if $L = \displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$.

Application: If $a_n \underset{n\to\infty}{\not\to} 0$ we know the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges (Ex: $\displaystyle\sum_{n=1}^{\infty} (-1)^n$)

Definition If $a_n \geq 0$ for n large enough we say $\sum_{n=1}^{\infty} a_n$ is a <u>series with positive terms</u>.

Note: In this situation $S_n = a_1 + \cdots + a_n$ is increasing (for n large enough)

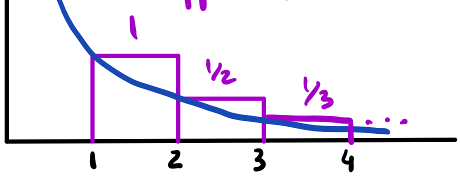
So we know a convergence criterion for it! (see Lecture 46)

Proposition 2: Assume $\sum_{n=1}^{\infty} a_n$ is a series with positive terms. Then, the series converges if and only if the partial sums $\{S_n\}_n$ form a bounded sequence.

Application The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \geq \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) - \ln(1) = \ln(n+1) \xrightarrow[n\to\infty]{} \infty$$

(upper Riemann Sum)



So S_n is not bounded!

Q: Can we find alternative bounds?

A YES! Pick an integer m. Then,

Claim: $S_n > \frac{m+1}{2}$ $\quad n > 2^{m+1}$

Indeed if $n > 2^{m+1}$ we have $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} > S_{2^{m+1}} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{m+1}}$

(fewer Terms!)

Now, we regroup by powers of 2

$$S_{2^{m+1}} = \underbrace{\left(1 + \frac{1}{2}\right)}_{> 1/2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> 2 \cdot \frac{1}{4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> 4 \cdot \frac{1}{8}} + \cdots + \underbrace{\left(\frac{1}{2^m + 1} + \cdots \frac{1}{2^{m+1}}\right)}_{2^m \frac{1}{2^{m+1}}}$$

$$> \underbrace{\frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^m}{2^{m+1}}}_{m+1 \text{ summands}} = m+1 \cdot \frac{1}{2} = \frac{m+1}{2}$$

Since m is arbitrary, we conclude that S_n is unbounded!

Conclusion: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges EVEN THOUGH $\frac{1}{n} \xrightarrow[n\to\infty]{} 0$.

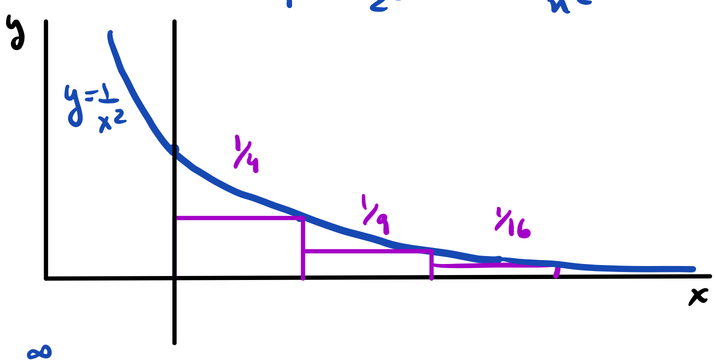
Natural Question: At what rate should $a_n \xrightarrow[n\to\infty]{} 0$ to ensure $\sum_{n=1}^{\infty} a_n$ converges?
($\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges)

EXAMPLE 2 $a_n = \frac{1}{n^2} \rightsquigarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$ (Euler)

To show convergence it's enough to bound the partial sums because $a_n \geq 0$ for all n.
We'll show this in 2 ways:

Method 1 $a_n = \frac{1}{n^2} = f(n)$ with $f(x) = \frac{1}{x^2}$

View $S_n = \frac{1}{1} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ as a Lower Riemann Sum (+ a constant)



$$S_n = 1 + \underbrace{\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2}}_{\leq \int_1^n \frac{1}{x^2}\,dx \,=\, \frac{-1}{x}\Big|_1^n \,=\, 1 - \frac{1}{n}}$$

So $S_n \leq 1 + 1 - \frac{1}{n} = 2 - \frac{1}{n} < 2$ for all n

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a bounded series with positive terms, so it converges!
And $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$. (limit is L.U.B. of partial sums)

Method 2 Work out a bound as we did with harmonic series:

$$S_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \cdots + \frac{1}{n\cdot n} \underset{\substack{1<2\\2<3\\ \vdots \\ n-1<n}}{<} 1 + \underbrace{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)\cdot n}}_{\text{Looks like the telescopic example}}$$

$$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 + \left(1 - \frac{1}{n}\right) = 2 - \frac{1}{n}$$

Conclusion: The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but the sum of <u>some</u> of its terms ($= \frac{1}{n^2}$) converges!

EXAMPLE 3: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges (to $\ln 2$)
This is an example of a (sign) alternating series with $a_n \xrightarrow[n\to\infty]{} 0$ (next time!)