Lecture XLIX: §13.4 (cont) General Properties
§13.6 The Integral Test

§1 General Convergence Criteria:

Recall: $\sum_{n=1}^{\infty} a_n$ series with positive terms ($a_n \geq 0$ for n large enough) is convergent if and only if it is bounded

EXAMPLE 1: $\sum_{n=0}^{\infty} \frac{1}{n!} = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots = e$ ($0! = 1! = 1$ by definition)

We check it is bounded (all $a_n = \frac{1}{n!} > 0$)

$S_0 = 1$

$S_1 = 1+1$

$S_2 = 1+1+\frac{1}{2!}$

$S_3 = 1+1+\frac{1}{2}+\frac{1}{2\cdot 3}$

$\leadsto$ $S_n = 1+1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{2\cdot 3\cdot\cdots\cdot n}$ for $n \geq 3$

$$\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}$$

$$= 1+\sum_{k=0}^{n-1}\frac{1}{2^k} < 1+\sum_{k=0}^{\infty}\frac{1}{2^k} = 1+\frac{1}{1-\frac{1}{2}} = 3.$$

Conclusion $S_n < 3$ for all $n \geq 3$, so the series converges & its sum is ≤ 3.

- We'll see that $\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x$ for all x, so taking $x=1$ yields $\sum_{n=1}^{\infty}\frac{1}{n!} = e$.
- We can use this to show e is not rational (See the end of this notes)

Q: What is the effect of rearranging sums?

A: We have to be very careful!

EXAMPLE $\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+\cdots$ diverges $(-1)^n \underset{n\to\infty}{\not\to} 0$

But, if we group consecutive terms we will converge.

- $\sum_{n=0}^{\infty} (-1)^{2n}+(-1)^{2n+1} = (1-1)+(1-1)+\cdots = 0$
- $1+\sum_{n=0}^{\infty} (-1)^{2n+1}+(-1)^{2n} = 1+(-1+1)+(-1+1)+\cdots = 1$

Conclusion: Rearranging the sum by inserting parenthesis as above can change the value of the series if it was divergent. However, such rearrangement will give the same sum if $\sum_{n=1}^{\infty} |a_n|$ converges (see Appendix A13)

(Name: absolutely convergent)

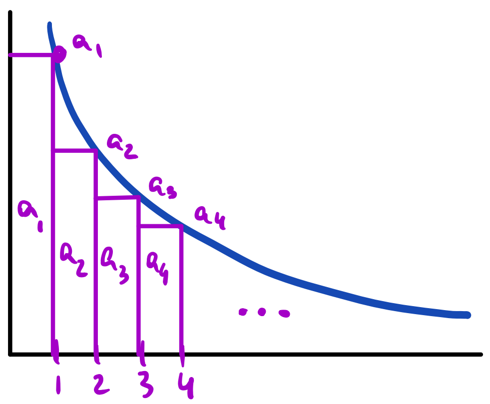
§2. The Integral Test:

GOAL: Use improper integrals to test for convergence of $\sum_{n=1}^{\infty} a_n$.
Assume $a_n \geq 0$ for all n & $(a_n)_n$ decreasing.

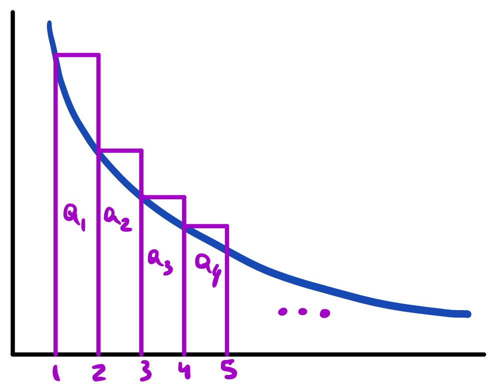
Q: What functions can we use?

- Need $f: [1,\infty) \longrightarrow \mathbb{R}$ with $f(n) = a_n$ & f continuous (so integrable!)
- $f(x) \geq 0$ for all x & $f(x)$ decreasing ($f' \leq 0$ is enough)

IDEA: Relate $\int_1^{\infty} f(x)\,dx$ to the series of lower & upper Riemann sums.



LOWER R.S.



UPPER R.S.

f decreasing and $f(x) \geq 0$ gives

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx \leq a_1 + \cdots + a_{n-1}$$

So
$$\boxed{\sum_{j=1}^{n} a_j} \underset{\text{add } a_1 \geq 0}{\leq} a_1 + \int_1^n f(x)\,dx \underset{\text{add } \int_n^{n+1} f(x)\,dx \geq 0}{\leq} a_1 + \int_1^{n+1} f(x)\,dx \leq a_1 + \boxed{\sum_{j=1}^{n} a_j}$$

Conclude: $\sum_{j=1}^{\infty} a_j$ & $\int_1^{\infty} f(x)\,dx$ both converge or diverge. (they share the same behavior)

Cauchy Integral Test: If $f : [1,\infty) \to \mathbb{R}_{\geq 0}$ is decreasing & $a_n = f(n)$ for all n, then either $\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

Why? Our earlier comparisons give:

$$(*) \quad \boxed{\sum_{j=1}^{n} a_j} \leq a_1 + \boxed{\int_1^n f(x)\,dx} \leq a_1 + \boxed{\sum_{j=1}^{n} a_j}$$

Note $S_n = \sum_{j=1}^{n} a_j$ is increasing because $a_n \geq 0$ for all n.
$b_n = \int_1^n f(x)\,dx$ ——— $f(x) \geq 0$ $\left(b_{n+1} - b_n = \int_n^{n+1} f(x)\,dx \geq 0\right)$

So both sequence converge if they are bounded & diverge otherwise.

But $(*)$ says they are both bounded above or neither.

- A bound B for S_n gives a bound B for b_n
- A bound B " b_n ——— $a_1 + B$ for S_n.

Conclusion S_n & b_n are both convergent or divergent.

By construction $\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx = \lim_{n\to\infty} \int_1^n f(x)\,dx$
$f(x) \geq 0$

Examples ① $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because $\int_1^n \frac{1}{x}\,dx = \ln n - \ln 1 \xrightarrow[n\to\infty]{} \infty$

Use $f(x) = \frac{1}{x}$ positive, decreasing & continuous

② $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because $\int_1^n \frac{dx}{x^2} = \frac{-1}{x}\Big|_1^n = 1 - \frac{1}{n} \xrightarrow[n\to\infty]{} 1$ converges

Use $f(x) = \frac{1}{x^2}$: positive, decreasing & continuous.

③ In general: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (p-series) converges if and only if $p > 1$.

Why? If $p \leq 0$, then $\frac{1}{n^p} \underset{n\to\infty}{\not\longrightarrow} 0$ so the p-series diverges

- We focus on $p > 0$ with $p \neq 1$ ($p = 1$ is EXAMPLE ①)

Take $f(x) = \frac{1}{x^p}$ on $[1,\infty)$

- f is continuous, positive

- $f' = \frac{-p}{x^{p+1}} < 0$ on $(1,\infty)$ so f is decreasing.

$$\int_1^n f(x)\,dx = \int_1^n \frac{1}{x^p}\,dx = \int_1^n x^{-p}\,dx \underset{p\neq 1}{=} \frac{x^{-p+1}}{-p+1}\Big|_1^n = \frac{1}{1-p}\left(n^{1-p}-1\right)$$

& $$\lim_{n\to\infty} \frac{n^{1-p}-1}{1-p} = \begin{cases} \infty & \text{if } p<1 \\ \frac{0-1}{1-p} & \text{if } p>1 \end{cases}$$

Conclusion: The p-series converges if and only if $p>1$ (by the Integral Test)

§3. Optional: e is irrational

Our starting point is $e = 1+1+\frac{1}{2!}+\dots+\frac{1}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}$

This gives $e-1-1-\frac{1}{2!}-\dots-\frac{1}{n!} = \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\dots\dots \geq 0$ for each n.

To show that e is not rational, we argue by contradiction. Write $e=\frac{p}{q}$ with p,q integers (so e is rational) Pick n large enough so that $n>q$ & define $a = n!\left(e-1-1-\frac{1}{2!}-\dots-\frac{1}{n!}\right)$

- Note $q \mid n!$ so $n!e$ is an integer. This implies a is also a positive integer.

But $a = n!\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\dots\dots\right) = \frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\dots$

$$< \frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\dots = \frac{1}{n+1}\left(1+\frac{1}{n+1}+\frac{1}{(n+1)^2}+\dots\right)$$

$$= \frac{1}{n+1}\sum_{k=0}^{\infty}\left(\frac{1}{n+1}\right)^k \underset{\text{Geometric Series}}{=} \frac{1}{n+1}\cdot\frac{1}{1-\frac{1}{n+1}} = \frac{1}{n}$$

So $0<a<\frac{1}{n}$ & a is an integer. This cannot happen!

We conclude from this that e cannot be rational.