Lecture L: §13.6 The Integral Test. Euler's Constant
§13.7 The Ratio Test & the Root Test

§1 The integral Test:

Cauchy Integral Test: If $f : [1,\infty) \to \mathbb{R}_{\geq 0}$ is decreasing & $a_n = f(n)$ for all n, then either $\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x)\,dx$ both converge or both diverge

EXAMPLES ① $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n} = ?$ $\qquad f(x) = \dfrac{1}{x \ln x}$ defined on $[2, \infty)$

- f is continuous & positive
- $f'(x) = \dfrac{-1}{(x\ln x)^2} \underbrace{(\ln x + 1)}_{>1 \ (x \geq 2)} < 0 \leadsto f$ is decreasing

Test says $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges if and only if $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\,dx$ does.

$$\int_2^n \frac{dx}{x \ln x} \underset{\substack{u = \ln x \\ du = \frac{dx}{x}}}{=} \int_{\ln 2}^{\ln n} \frac{du}{u} = \ln u \Big|_{\ln 2}^{\ln n} = \underbrace{\ln(\underbrace{\ln n}_{\downarrow n\to\infty \ \infty}) - \ln(\ln 2)}_{\downarrow n\to\infty \ \infty} \xrightarrow[n\to\infty]{} \infty$$

Conclusion: The series diverges.

② $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} = ??$ with $p \geq 0$

Note $(\ln n)^p \leq \ln n$ for $p \leq 1$ & $n \geq 3$ (because $\ln n \geq 1$)

So $\dfrac{1}{n(\ln n)^p} \geq \dfrac{1}{n \ln n}$. Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges, (by ①) so does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ $(p \leq 1)$

(This is an example of Comparison Theorems, which we'll see next time)

- If $p > 1$, we use the Integral Test with $f(x) = \dfrac{1}{x(\ln x)^p}$ on $[2, \infty)$
- f is continuous & positive
- $f'(x) = \dfrac{-1}{(x(\ln x)^p)^2}\left((\ln x)^p + p(\ln x)^{p-1}\right) = \dfrac{-(\ln x)^{p-1}}{x^2 (\ln x)^{2p}} \underbrace{(p + \ln x)}_{>0} > 0 \quad (x \geq 2)$

So f is decreasing.

$$\int_2^n \frac{dx}{x(\ln x)^p} \underset{u = \ln x}{=} \int_{\ln 2}^{\ln n} u^{-p}\,du \underset{p \neq 1}{=} \frac{u^{-p+1}}{-p+1} \Bigg|_{\ln 2}^{\ln n} = \frac{1}{1-p}\left(\frac{1}{\underbrace{(\ln n)^{p-1}}_{\xrightarrow[n\to\infty]{} \infty \ (p>1)}} - \frac{1}{(\ln 2)^{p-1}}\right) \longrightarrow \frac{1}{(p-1)(\ln 2)^{p-1}}$$

Conclusion: The series converges for $p>1$ & sum $\leq a_2 + \int_2^\infty f(x)\,dx$

$$= \frac{1}{2(\ln 2)^p} + \frac{1}{(p-1)(\ln 2)^{p-1}}.$$

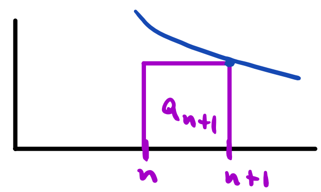
The series diverges for $0 \leq p \leq 1$.

Q: What else can we learn from the Test?

Key for the integral Test: Our assumptions on f give

$$\boxed{\sum_{j=1}^{n} a_j} \leq a_1 + \boxed{\int_1^n f(x)\,dx} \leq a_1 + \boxed{\sum_{j=1}^{n} a_j}$$

$$\rightsquigarrow \quad 0 \leq \underbrace{\sum_{j=1}^{n} a_j - \int_1^n f(x)\,dx}_{=:F_n} \leq a_1$$



We have
- $\{F_n\}_n$ is bounded
- $\{F_n\}_n$ is decreasing : $F_{n+1} = F_n + \underbrace{a_{n+1} - \int_n^{n+1} f(x)\,dx}_{\leq 0} \leq F_n$

Conclusion : $\{F_n\}$ converges! Write $L = \lim_{n\to\infty} F_n$.

We have $\quad 0 \leq L = \lim_{n\to\infty} \left((a_1 + a_2 + \cdots + a_n) - \int_1^n f(x)\,dx \right) \leq a_1$.

§2 Application: Euler's Constant

Pick $a_n = \frac{1}{n}$ $\quad f(x) = \frac{1}{x}$ $\quad \rightsquigarrow \int_1^n \frac{dx}{x} = \ln n$

Write $\quad L = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \right)$ $\quad$ with $0 \leq L \leq a_1 = 1$.

Equivalently $\quad \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - (\ln n + L) \right) = 0 \qquad (*)$

Name: $L = \gamma$ = Euler's Constant $\simeq 0.5772156649015328606\ldots$

Open Question: Is γ rational or not?

Algorithmic notation $\quad b_n = o(1)$ $\quad$ if $\frac{b_n}{1} \xrightarrow[n\to\infty]{} 0$

We can write the limit $(*)$ as $\quad 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + \gamma + o(1)$

§3 The Ratio Test

Motivation: $\sum_{n=0}^{\infty} r^n = \begin{cases} \text{converges to } \frac{1}{1-r} & \text{if } 0 \leq r < 1 \\ \text{diverges} & \text{if } r \geq 1 \end{cases}$ for $r \geq 0$

For the geometric series, the ratio between succesive terms is the constant r

$$\frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{r^n} = r$$

Ratio Test: Pick a sequence $\{a_n\}_n$ with $a_n > 0$ for n large enough

Assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ exists. Then:

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \text{converges} & \text{if } L < 1 \\ \text{diverges} & \text{if } L > 1 \\ \text{don't know} & \text{if } L = 1 \end{cases}$$ (anything can happen!)

Proof Future Lecture.

EXAMPLES ① $\sum_{k=0}^{\infty} \frac{1}{k!}$ converges (to e)

- Can show convergence by careful bounding techniques (Lecture 49)
- Alternative: use Ratio Test!

$$\frac{a_{n+1}}{a_n} = \frac{1/(n+1)!}{1/n!} = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$$ so the series converges!

② $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges (to e^3)

Ratio Test: $\frac{a_{n+1}}{a_n} = \frac{3^{n+1}/(n+1)!}{3^n/n!} = \frac{3}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$

③ $\sum_{n=0}^{\infty} \frac{n^6}{3^n}$ converges

Ratio Test $\frac{a_{n+1}}{a_n} = \frac{(n+1)^6/3^{n+1}}{n^6/3^n} = \left(\frac{n+1}{n}\right)^6 \frac{1}{3} \xrightarrow[n\to\infty]{} \frac{1}{3} < 1$

Remark This test is useful if the series involves factorials, powers, exponentials & products in general.

EXAMPLE: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and in both cases the ratio $\frac{a_n}{a_{n+1}}$ has limit $=1$, so the Ratio Test is inconclusive.

§4. The Root Test:

Motivation comes from geometric series. This Test will be extremely useful for deciding convergence of power series (e.g. Taylor series)

Root Test: Pick a sequence $\{a_n\}_n$ with $a_n > 0$ for n large enough.

Assume $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ exists. Then

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \text{converges} & \text{if } L<1 \\ \text{diverges} & \text{if } L>1 \\ \text{don't know} & \text{if } L=1 \end{cases}$$ (anything can happen!)

Proof Future Lecture.

EXAMPLES: ① $\sum_{n=1}^{\infty} \frac{1}{k}$ diverges & $\sqrt[k]{\frac{1}{k}} = \frac{1}{k^{1/k}} \xrightarrow[k\to\infty]{} \frac{1}{1} = 1$. Test does not help

(because: $\ln(k)^{1/k} = \frac{1}{k}\ln(k) \xrightarrow[k\to\infty]{} 0$ so $k^{1/k} \longrightarrow e^0 = 1$)

② $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges & $\sqrt[k]{\frac{1}{k^2}} = \left(\sqrt[k]{\frac{1}{k}}\right)^2 \longrightarrow 1^2 = 1$ Test does not help us.

③ $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by Root Test $\sqrt[n]{\frac{1}{n^n}} = \frac{1}{n} \xrightarrow[n\to\infty]{} 0 < 1$

④ $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ ———————— $\sqrt[n]{\frac{1}{(\ln n)^n}} = \frac{1}{\ln n} \xrightarrow[n\to\infty]{} 0 < 1$

⑤ $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ ———————— $\sqrt[n]{\frac{n^2}{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \longrightarrow \frac{1^2}{2} = \frac{1}{2} < 1$

⑥ $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^{n/2}$ ———————— $\sqrt[n]{(\sqrt[n]{n} - 1)^{n/2}} = \sqrt[2]{\sqrt[n]{n} - 1} \xrightarrow[n\to\infty]{} \sqrt{1-1} = 0 < 1$

Remark: The Root Test is useful for series involving exponentials & powers.