Lecture L1: §13.5 Series of Non-negative terms. Comparison Tests

Comparison Test 1 Fix 2 sequences of non-negative terms $\{a_n\}_n$ & $\{b_n\}_n$.

Assume $0 \leq a_n < b_n$ for all n

Then: (1) If $\sum_{n=1}^{\infty} b_n$ converges with sum L, then $\sum_{n=1}^{\infty} a_n$ converges with sum $\leq L$.

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

EXAMPLES: ① $\frac{1}{n} < \frac{1}{\ln n}$ for all $n > 1$ ($y = \ln n$ is always below the line $y = x$)

so $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges because $\sum_{n=2}^{\infty} \frac{1}{n}$ does

② $\sum_{n=0}^{\infty} \frac{1}{3^n+1}$ converges because $\sum_{n=1}^{\infty} \frac{1}{3^n}$ does & $\frac{1}{3^n+1} \leq \frac{1}{3^n}$

Furthermore $\sum_{n=0}^{\infty} \frac{1}{3^n+1} \leq \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-1/3} = \frac{3}{2}$

Why does Comparison Test 1 work? Assume $0 \leq a_n \leq b_n$ for all n & take partial sums on both sides

$$0 \leq S_n = a_1 + \cdots + a_n \leq \tilde{S}_n = b_1 + \cdots + b_n$$

Since S_n & $\tilde{S}_n$ are both increasing they converge if they are bounded above & diverge otherwise.

(1) If $\tilde{S}_n \longrightarrow L$, then $\tilde{S}_n \leq L$ for n large enough (L = least upper bound)

Then $S_n \leq \tilde{S}_n \leq L$ for n large enough, so S_n is bounded & it converges to L' with $L' \leq L$.

(2) If S_n diverges, then its limit is ∞ (because its increasing & unbounded)

Then $\tilde{S}_n$ also has $\lim_{n\to\infty} \tilde{S}_n = \infty$ so $\sum_{n=1}^{\infty} b_n$ diverges.

Obs: We can extend the result if the conditions hold for n large enough (say $n \geq n_0$), by rewriting both series by separating the first n_0 terms:

$$\sum_{n=1}^{\infty} a_n = a_1 + \cdots + a_{n_0-1} + \boxed{\sum_{n=n_0}^{\infty} a_n}$$

$$\sum_{n=1}^{\infty} b_n = b_1 + \cdots + b_{n_0-1} + \boxed{\sum_{n=n_0}^{\infty} b_n}$$

→ comparison Test holds for this part (called the tail), so it holds for the whole series.

In this setting $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n + (a_1 - b_1) + \cdots + (a_{n_0-1} - b_{n_0-1})$.

EXAMPLE $\sum_{n=1}^{\infty} \frac{n+1}{n^n}$ converges by comparing it to $\frac{2}{n^2}$.

How? $\frac{n+1}{n^n} = \frac{1}{n^{n-1}} + \frac{1}{n^n} < \frac{2}{n^{n-1}} \leq \frac{2}{n^2}$ for $n \geq 3$ $(n-1 \geq 2)$

so for $n \geq 3$: $\sum_{n=3}^{\infty} \frac{n+1}{n^n} \leq \sum_{n=3}^{\infty} \frac{2}{n^3}$ converges so $\sum_{n=3}^{\infty} \frac{n+1}{n^n}$ converges too.

Add the first 2 terms to get $\sum_{n=1}^{\infty} \frac{n+1}{n^n} \leq \sum_{n=1}^{\infty} \frac{2}{n^3} + \left(\frac{2}{1} - \frac{2}{1}\right) + \left(\frac{3}{4} - \frac{2}{4}\right)$

Limit Comparison Test: Suppose $\{a_n\}, \{b_n\}$ are non-negative sequences with $b_n > 0$ for n large enough. Assume $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$. Then, either both $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ converge or they both diverge.

Why? Since $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then for $\varepsilon = \frac{L}{2}$ we can find $n_0 > 0$ for which $\frac{L}{2} = L - \varepsilon < \frac{a_n}{b_n} < L + \varepsilon = \frac{3L}{2}$ for $n \geq n_0$.

Since $b_n > 0$ we get $\boxed{0 \leq b_n \frac{L}{2} \overset{(1)}{<} a_n \overset{(2)}{<} \frac{3L}{2} b_n}$

- So if $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{n=1}^{\infty} \frac{b_n L}{2} = \frac{L}{2} \sum_{n=1}^{\infty} b_n$ by (1) + the Comparison Test. Since $L \neq 0$, we conclude that $\sum_{n=1}^{\infty} b_n$ also converges.
- Similarly, if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} \left(\frac{3L}{2}\right) b_n$. Again, by the Comparison Test applied to (2) we get that $\sum_{n=1}^{\infty} a_n$ converges.
- Same arguments + the inequalities (1) & (2) show that if one series diverges, then the other one does as well.

EXAMPLE: ① $\sum_{n=1}^{\infty} \frac{n+2}{2n^3 - 3}$ $\quad a_n = \frac{n+2}{2n^3-3} \geq 0$ & we know $b_n = \frac{1}{n^2}$

gives a convergent sequence.

Now: $\frac{a_n}{b_n} = \frac{n+2}{2n^3-3} \Big/ \frac{1}{n^2} = \frac{n^3+2n}{2n^3-3} = \frac{1+2/n^2}{2-3/n^3} \xrightarrow[n\to\infty]{} \frac{1}{2} > 0$

So by comparison, $\sum_{n=1}^{\infty} a_n$ converges.

<u>Observation</u>: Limit Comparison Test are useful when we compare to $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n!}$, $\sum_{n=1}^{\infty} x^n$ (for x fixed) (Typical choices for $\sum_{n=1}^{\infty} b_n$).

② (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ & diverges for $p \leq 1$ (Lecture 49)

- If $p \leq 1$, then $n^p \leq n$ so $\frac{1}{n^p} \geq \frac{1}{n}$. By Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$ because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- If $p \geq 2$ then $n^p \geq n^2$ so $\frac{1}{n^p} \leq \frac{1}{n^2}$ so by comparison, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p \geq 2$
- For $1<p<2$ we know it converges by the Cauchy Integral Test.

Let's use comparison instead:

Take $S_n = \sum_{k=1}^{n} \frac{1}{k^p}$ We will show that $S_n < \frac{2^{p-1}}{2^{p-1}-1}$

so $(S_n)_n$ is positive, increasing & bounded above, so it must converge.

Given n, pick m with $n < 2^m$. Then:

$$S_n = \sum_{k=1}^{n} \frac{1}{k^p} \leq \sum_{k=1}^{2^m-1} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2^m-1)^p}$$

we added more terms!

$$= 1 + \underbrace{\left(\boxed{\frac{1}{2^p}} + \frac{1}{3^p}\right)}_{2 \text{ terms}} + \underbrace{\left(\boxed{\frac{1}{4^p}} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)}_{4 \text{ terms}} + \cdots + \underbrace{\left(\boxed{\frac{1}{(2^{m-1})^p}} + \cdots + \frac{1}{(2^m-1)^p}\right)}_{2^{m-1} \text{ terms}}$$

group by powers of 2

$\boxed{}$ is the largest terms in each group so:

$$S_n < 1 + \frac{1}{2^p}\cdot 2 + \frac{1}{4^p}\cdot 4 + \cdots + \frac{1}{(2^{m-1})^p}\cdot 2^{m-1}.$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \cdots + \frac{1}{(2^{m-1})^{p-1}}$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \cdots + \frac{1}{(2^{p-1})^{m-1}}$$

$$= \sum_{k=0}^{m-1} \frac{1}{(2^{p-1})^k} \underset{\text{Geom series}}{\le} \sum_{k=0}^{\infty} \frac{1}{(2^{p-1})^k} = \frac{1}{1-\frac{1}{2^{p-1}}} = \frac{2^{p-1}}{2^{p-1}-1}.$$

$\frac{1}{2^{p-1}} < 1$ because $p > 1$

We got the bound for S_n we were looking for.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}+\sqrt{n}}{n}$ diverges

Take $a_n = \dfrac{\sqrt{n+1}+\sqrt{n}}{n}$ & $b_n = \dfrac{\sqrt{n}}{n} = \dfrac{1}{n^{1/2}}$

$$\frac{a_n}{b_n} = \frac{(\sqrt{n+1}+\sqrt{n})/n}{\sqrt{n}\;/n} = \sqrt{\frac{n+1}{n}} + 1 \xrightarrow[n\to\infty]{} \sqrt{1}+1 = 2 \neq 0$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, by the Limit Comparison Test $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}+\sqrt{n}}{n}$ diverges as well.

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ converges

__Idea__: $a_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{n}\cdot\dfrac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \dfrac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})} = \dfrac{1}{n(\sqrt{n+1}+\sqrt{n})}$

Take $b_n = \dfrac{1}{n\sqrt{n}} = \dfrac{1}{n^{3/2}}$

$$\frac{a_n}{b_n} = \frac{\frac{1}{n(\sqrt{n+1}+\sqrt{n})}}{\frac{1}{n\sqrt{n}}} = \frac{n\sqrt{n}}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{1}+1} = \frac{1}{2} \neq 0$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges (p-series with $p=\frac{3}{2}>1$), by the Limit Comparison Test, so does $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$.

<u>Consequence</u>: Rearranging a convergent non-negative sequence does <u>not</u> change the sum.

<u>Why?</u> Say we rearrange to b_n: (eg: $b_1 = a_{10}$, $b_2 = a_3$, $b_3 = a_1$, $b_4 = a_{200}$ ...)

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots = S$$

Call S_n the partial sums

$$\sum_{n=1}^{\infty} b_n = a_{10} + a_3 + a_1 + a_{200} + \cdots$$

Call B_n ______

We see $B_4 = b_1 + b_2 + b_3 + b_4 = a_{10} + a_3 + a_1 + a_{200} \underset{a_n \geq 0}{\leq} a_1 + a_2 + \cdots + a_{200} = S_{200}$

So $B_4 \leq S_{200} \leq \sum_{n=1}^{\infty} a_n = S$.

- Same idea works in general: $B_n \leq S$ for all n

  But (B_n) are increasing, so they converge & their limit is $\leq S$.

- Say $\sum_{n=1}^{\infty} b_n = B$. We know $B \leq S$.

- But (a_n) is a rearrangement of (b_n), so by the same argument we see that $\sum_{n=1}^{\infty} a_n = S \leq B$ (switching the roles of (a_n), (b_n), S & B)

<u>Conclude</u>: $S \leq B$ & $B \leq S$ so $\boxed{S = B}$