Lecture L11: §13.5 Comparison Tests
§13.7 Ratio & Root Tests
§13.8 The Alternating series Test

Comparison Test 1 Fix 2 sequences of non-negative terms $\{a_n\}_n$ & $\{b_n\}_n$.

Assume $0 \leq a_n < b_n$ for all n large enough

Then: (1) If $\sum_{n=1}^{\infty} b_n$ converges with sum L, then $\sum_{n=1}^{\infty} a_n$ converges

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

Limit Comparison Test: Suppose $\{a_n\}, \{b_n\}$ are non-negative sequences with $b_n > 0$ for n large enough. Assume $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$. Then, either both $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ converge or they both diverge. (same behavior!)

§1. Application 1: Proof of Ratio & Root Test

Ratio & Root Tests: Pick $\{a_n\}_n$ a sequence of positive integers (for n large enough). Assume either $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = \Pi$ ($a_n > 0$ for n large enough) exist. Then:

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \text{converges if} & L<1 & \Big| & \Pi<1 \\ \text{diverges if} & L>1 & \Big| & \Pi>1 \\ \text{don't know if} & L=1 & \Big| & \Pi=1 \\ & \text{RATIO TEST} & & \text{ROOT TEST} \end{cases}$$

(anything can happen!)

Observation Appendix A12 has a refinement if $L=1$ or $\Pi=1$ & the limit comes from below or above 1.

EXAMPLES:

① $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ converges by Ratio Test $\frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} = \left(\frac{n+1}{n}\right)^2 \frac{1}{2} \longrightarrow \frac{1}{2} < 1$

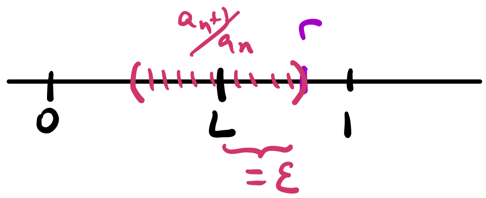
(Alt: Root Test $\sqrt[n]{\frac{n^2}{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \longrightarrow \frac{1^2}{2} = \frac{1}{2} < 1$

($\sqrt[n]{n} \to 1$ because $\ln \sqrt[n]{n} = \frac{\ln n}{n} \longrightarrow 0$ by L'Hôpital)

② $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converges by Root Test $\sqrt[n]{\frac{1}{(\ln n)^n}} = \frac{1}{\ln n} \xrightarrow[n\to\infty]{} 0 < 1$

Proof We use Comparison Test with the geometric series $\sum_{n=0}^{\infty} r^n$ (or a tail of it) for a suitable $r > 0$

Ratio Test: (1) Suppose $L < 1$ & pick r with $L < r < 1$ & set $\varepsilon = r - L$.



Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, we can find n_0 so that

$$L - \varepsilon < \frac{a_{n+1}}{a_n} < L + \varepsilon = r \quad \text{for } n \geq n_0$$

So
$$a_{n_0+1} < r\, a_{n_0}$$
$$a_{n_0+2} < r\, a_{n_0+1} < r^2 a_{n_0}$$
$$a_{n_0+3} < r\, a_{n_0+2} < r^3 a_{n_0}$$
$$\vdots$$
$$a_{n_0+k} < r^k a_{n_0} \quad \text{for all } k \geq 1$$

So Tail $= \sum_{m=n_0}^{\infty} a_m \leq \sum_{m=n_0}^{\infty} r^m a_{m_0} =$

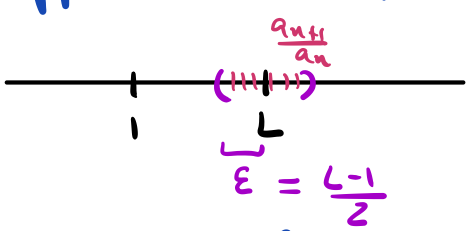
$$= a_{n_0} \sum_{m=n_0}^{\infty} r^m = a_{n_0} \frac{r^{n_0}}{1-r}$$

$r < 1$

By Comparison Test 1: $\sum_{m=n_0}^{\infty} a_m$ converges, so $\sum_{m=1}^{\infty} a_m$ also converges.

(Tail)

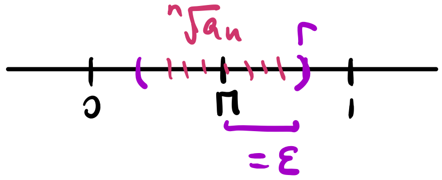
(2) Suppose $L > 1$ then $a_{n+1} > a_n > 0$ for all n large enough
$n \geq n_0$



So $a_n \underset{n\to\infty}{\not\to} 0$ $\quad (a_n \geq a_{n_0}$ for $n \geq n_0)$

This ensures $\sum_{n=1}^{\infty} a_n$ diverges.

Root Test (1) Suppose $M < 1$ & pick r with $M < r < 1$ & set $\varepsilon = r - M > 0$



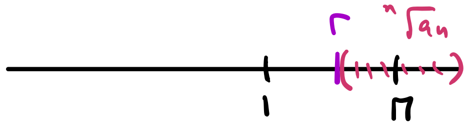
Since $\lim_{n\to\infty} \sqrt[n]{a_n} = M$ we can find $n_0 > 0$ so that $M - \varepsilon < \sqrt[n]{a_n} < M + \varepsilon = r$ for $n \geq n_0$.

In particular: $a_n < r^n$ for $n \geq n_0$.

Then $\sum_{n=n_0}^{\infty} a_n \leq \sum_{n=n_0}^{\infty} r^n = \frac{r^{n_0}}{1-r}$ because $0<r<1$

Again, by Comparison Test 1, $\underbrace{\sum_{n=n_0}^{\infty} a_n}_{(\text{tail})}$ converges so $\sum_{n=1}^{\infty} a_n$ also converges

(2) Now assume $M>1$ & pick r with $M>r>1$. Set $\varepsilon = M-r>0$



Since $\lim_{n\to\infty} \sqrt[n]{a_n} = M$ we can find $n_0>0$

so that $r = M-\varepsilon < \sqrt[n]{a_n} < M+\varepsilon$ if $n \geq n_0$.

Then $r^n < a_n$ for all $n \geq n_0$, giving $\underbrace{\sum_{n=n_0}^{\infty} r^n}_{\text{diverges because } r>1.} \leq \sum_{n=n_0}^{\infty} a_n$

Again, the Comparison Test 1 says $\underbrace{\sum_{n=n_0}^{\infty} a_n}_{(\text{tail})}$ must diverge. Then, the original series $\sum_{n=1}^{\infty} a_n$ also diverges

§2 Alternating Series:

Q: What about series that don't have constant sign?

Particular case: sign alternates $+-+-+\cdots$ or $-+-+\cdots$

Convenient notation: $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$ with $a_k \geq 0$.

EXAMPLES (1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \; (= \ln 2)$

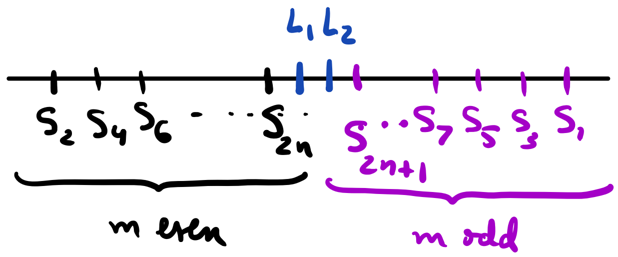
(2) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots\cdots$

Note The alternation of sign gives cancellations which in general help for convergence.

GOAL: Decide convergence / divergence of these series. HARDER: Compute their sum.

EXAMPLE $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$

Main strategy: Use $a_n = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$ & decreasing!



Look at partial sums with even and odd indices.

$S_1 = 1 > 0$

$S_2 = 1 - \frac{1}{2} = \frac{1}{2} < S_1$

$S_3 = S_2 + \frac{1}{3} > S_2$ & $S_3 = S_1 + \underbrace{\frac{1}{3} - \frac{1}{2}}_{<0} < S_1$

$S_4 = S_3 - \frac{1}{4} < S_3$ & $S_4 = S_2 + \underbrace{\frac{1}{3} - \frac{1}{4}}_{>0} > S_2$

Continuing in this way we get

$$S_{2(n+1)} = S_{2n} + \frac{(-1)^{2n+1+1}}{2n+1} + \frac{(-1)^{2n+2+1}}{2n+2} = S_{2n} + \underbrace{\frac{1}{2n+1} - \frac{1}{2n+2}}_{>0} > S_{2n}$$

$$S_{2(n+1)} = S_{2n+1} - \frac{1}{2n+2} < S_{2n+1}$$

So $\boxed{S_{2n+1} > S_{2(n+1)} > S_{2n} \quad \text{for all } n \geq 1}$

Similarly
$$S_{2(n+1)+1} = S_{2n+1} + \frac{(-1)^{2n+2+1}}{2n+2} + \frac{(-1)^{2n+3+1}}{2n+3} = S_{2n+1} \underbrace{- \frac{1}{2n+2} + \frac{1}{2n+3}}_{<0} < S_{2n+1}$$

$$S_{2(n+1)+1} = S_{2n+2} + \frac{1}{2n+3} > S_{2n+2} = S_{2(n+1)}$$

So $\boxed{S_{2n+1} > S_{2(n+1)+1} > S_{2(n+1)} \quad \text{for all } n \geq 1}$

We get:

(1) $S_1 > S_3 > S_5 > S_7 > \cdots > S_{2m+1}$ (ODD indices)

(2) $S_2 < S_4 < S_6 < S_8 < \cdots < S_{2m}$ (EVEN indices)

(3) In addition: $S_{2m} < S_{2m+1} \leq S_1 \rightsquigarrow \{S_{2m}\}_m$ is increasing & bounded above

$S_{2m+1} > S_{2m} \geq S_2 \rightsquigarrow \{S_{2m+1}\}_m$ is decreasing & bounded below

- $\{S_{2m}\}_m$ & $\{S_{2m+1}\}_m$ both converge & $|S_{2n+1} - S_{2n}| = \left|\frac{1}{2n+1}\right| \xrightarrow[n\to\infty]{} 0$ so they have the same limit! This limit is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. (next we show it's $\ln 2$).

§3 Application: Euler's Constant

Pick $a_n = \frac{1}{n}$ $\quad f(x) = \frac{1}{x}$ $\quad \rightsquigarrow \int_1^n \frac{dx}{x} = \ln n$

Write $L = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$ with $0 \leq L \leq a_1 = 1$.

Equivalently $\lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - (\ln n + L)\right) = 0$ $\quad (*)$

Name: $L = \gamma$ = Euler's Constant $\simeq 0.5772156649015328606\ldots$

Algorithmic notation: $b_n = o(1)$ if $\frac{b_n}{1} \xrightarrow[n\to\infty]{} 0$

We can write the limit (*) as $1+\frac{1}{2}+\cdots+\frac{1}{n} = \ln n + \gamma + o(1)$

Consequence: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

Why? Using the even & odd partial sums we know $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. (page 1)

- Now, look closer at the partial sums S_{2n} with even indices

$$S_{2n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}$$

regroup

$$= \underbrace{\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)}_{\text{ODD denominators}} - \underbrace{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)}_{\text{EVEN denominators}}$$

Add & substract even denom

$$= \left(1+\frac{1}{2}+\cdots\cdots+\frac{1}{2n}\right) - 2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)$$

$$= \left(1+\frac{1}{2}+\cdots+\frac{1}{2n}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

$$= \ln 2n + \gamma + o(1) - (\ln n + \gamma + o(1))$$

$$= \ln 2n - \ln n + o(1) = \ln\frac{2n}{n} + o(1) = \ln 2 + o(1)$$

Conclusion $\lim_{n\to\infty} S_{2n} = \ln 2$. Since $S_{2n+1} - S_{2n} = \frac{1}{2n+1} \to 0$, both have the same limit!

Since $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} = \ln 2$, we get $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$.

⚠ We have to be careful when resuming a series with alternating sums (unless $\sum_{n=1}^{\infty} |(-1)^{n+1} a_n|$ converges). More on this next time!