Lecture LIII: §13.8 The alternating series test. Absolute convergence

Up to now: We studied mostly convergence criteria for series of positive terms

Tests at hand:
- comparison
- limit comparison
- Integral Test
- Root / Ratio Test

Q: What about series that don't have constant sign?

Next: A criterion for alternating series (eg $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$)

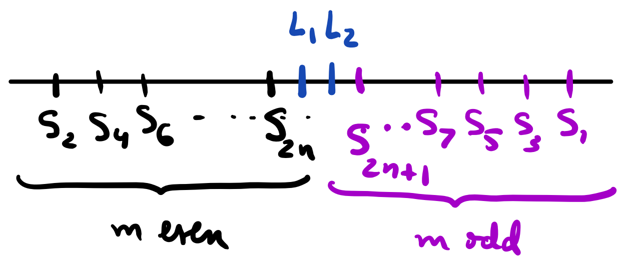
§1 Alternating Series:

Alternating Series Test: Pick an alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ with

1) $a_n \geq 0$ for all n
2) $\lim_{n \to \infty} a_n = 0$
3) $\{a_n\}_n$ is a decreasing sequence. ($a_1 \geq a_2 \geq a_3 \geq \cdots$)

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 \cdots$ converges

Why? Main strategy: Look at partial sums with even and odd indices.



$S_1 = \; > 0$

$S_2 = S_1 - a_2 < S_1$

$S_3 = S_2 + a_3 > S_2$

$S_3 = S_1 + \underbrace{a_3 - a_2}_{<0} < S_1$

$S_4 = S_3 - a_4 < S_3$

$S_4 = S_2 + \underbrace{a_3 - a_4}_{<0} > S_2$

Continuing in this way we get

$$S_{2(n+1)} = S_{2n} + (-1)^{2n+2} a_{2n+1} + (-1)^{2n+2+1} a_{2n+2} = S_{2n} + \underbrace{a_{2n+1} - a_{2n+2}}_{>0} > S_{2n}$$

$$S_{2(n+1)} = S_{2n+1} + (-1)^{2n+3} a_{2n+2} = S_{2n+1} - a_{2n+2} < S_{2n+1}$$

So $\boxed{S_{2n+1} > S_{2(n+1)} > S_{2n} \quad \text{for all } n \geq 1}$

Similarly $S_{2(n+1)+1} = S_{2n+1} + (-1)^{2n+2+1} a_{2n+2} + (-1)^{2n+3+1} a_{2n+3} = S_{2n+1} \underbrace{- a_{2n+2} + a_{2n+3}}_{<0} < S_{2n+1}$

$$S_{2(n+1)+1} = S_{2n+2} + (-1) a_{2n+3} > S_{2n+2} = S_{2(n+1)}$$

So $\boxed{S_{2n+1} > S_{2(n+1)+1} > S_{2(n+1)} \quad \text{for all } n \geq 1}$

We get:

(1) $S_1 > S_3 > S_5 > S_7 > \cdots > S_{2m+1}$ (ODD indices)

(2) $S_2 < S_4 < S_6 < S_8 < \cdots < S_{2m}$ (EVEN indices)

(3) In addition: $S_{2m} < S_{2m+1} \leq S_1$ ⇝ $\{S_{2m}\}_m$ is increasing & bounded above by S_1

$S_{2m+1} > \underset{\substack{\| \\ S_{2m+3}}}{S_{2(m+1)+1}} \geq S_2$ ⇝ $\{S_{2m+1}\}_m$ is decreasing & bounded below by S_2

Conclusion: Both sequences converge! Write $\lim_{m\to\infty} S_{2m} = L_2$

$S_{2k+1} = S_{2k} + (-1)^{2k+1+1} a_{2k+1}$ $\qquad \lim_{m\to\infty} S_{2m+1} = L_1$

But $\underset{\substack{\downarrow k\to\infty \\ |L_2 - L_1|}}{|S_{2k} - S_{2k+1}|} = |(-1)^{2k+2} a_{2k+1}| = a_{2k+1} \longrightarrow 0$ so $\boxed{L_1 = L_2}$

□

Observation: Same result works if (1) & (3) are true for n large enough.

EXAMPLES ① $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2+1}{5-k^2}$ alternating but diverges because $\displaystyle\frac{k^2+1}{5-k^2} \xrightarrow[k\to\infty]{} -1 \neq 0$

② $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\ln k}{k}$ alternating

- $a_n = \dfrac{\ln n}{n}$ is positive & $a_n \xrightarrow[n\to\infty]{} 0$ (L'Hôpital)
- $f(x) = \dfrac{\ln x}{x}$ is decreasing $f'(x) = \dfrac{\frac{1}{x}\cdot x - \ln x}{x^2} = \dfrac{1-\ln x}{x^2} < 0$ for $x > 1$

So (a_n) is decreasing.

The series converges by the Alternating Series Test.

Q: What if the terms change sings without alternating?

§2 Absolute Convergence:

Def: $\sum_{k=1}^{\infty} a_k$ is <u>absolutely convergent</u> if $\sum_{k=1}^{\infty} |a_k|$ converges.

Proposition: An absolutely convergent series always converges.

Why? Look at the series for $b_n = a_n + |a_n| \geq 0$.

- $0 \leq b_n \leq 2|a_n|$ & $\sum_{n=1}^{\infty} 2|a_n|$ converges, so $\sum_{n=1}^{\infty} b_n$ converges by comparison Test
- Now: $a_n = b_n - |a_n|$ & so by Limit Laws: we get
$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{\infty} b_n}_{\text{converges}} - \underbrace{\sum_{n=1}^{\infty} |a_n|}_{\text{converges}}$$
also converges.

Advantage: We have a lot of tests for absolute convergence!

Warning: The converse is false in general

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges (to $\ln 2$) but $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

So $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is NOT absolutely convergent.

Def: Series like these are called <u>conditionally convergent</u>.

Curious fact: If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then by rearranging we can make the new series converge to ANY prescribed value or even diverge (see Theorem 2 in Appendix A13).

Next, we show an example (optional reading)

EXAMPLE: $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ $\left(a_n = \frac{(-1)^{n+1}}{n}\right)$

- Multiply by $\frac{1}{2}$:

$$\frac{1}{2}\ln 2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12}$$

- Add the 2 series term-by-term as follows:

$$\begin{array}{rcl} \ln 2 &=& 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots \\ + \ \frac{\ln 2}{2} &=& \phantom{1 -{}} \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} + \cdots \end{array}$$

$$\frac{3\ln 2}{2} = 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} b_n$$

We see that this is a rearrangement(*) of the original series, but the sum is different! $\left(\ln 2 \neq \frac{3\ln 2}{2}\right)$

(*) • $b_{2k+1} = \frac{(-1)^{2k+2}}{2k+1} = \frac{1}{2k+1}$

• $b_{2k} = \frac{(-1)^{2k+1}}{2k} + (-1)^{k+1}\frac{1}{2k} = \frac{-1+(-1)^{k+1}}{2k} = \begin{cases} 0 & \text{if } k \text{ even} \\ -\frac{1}{k} & \text{if } k \text{ is odd} \end{cases}$

So the sequence $\{b_n\}$ has the same terms as a_n, but reordered.

- The odd terms a_{2k+1} appear as b_{2k+1}
- The even terms appear / disappear depending on $4 \mid n$ or not.

$a_{4m+2} = \frac{(-1)}{4m+2}$ disappears from $\frac{(-1)^{4m+1}}{4m+2} + \frac{(-1)^{(2m+1)+1}}{4m+2} = 0$ but

appears from $\frac{(-1)^{2(4m+2)+1}}{2(4m+2)} + \frac{(-1)^{2(4m+2)+1}}{2(4m+2)} = \frac{-2}{2(4m+2)} = \frac{-1}{4m+2} = \frac{(-1)^{4m+2+1}}{4m+2}$

$a_{4m} = \frac{(-1)}{4m}$ appears from $\frac{(-1)^{8m+1}}{8m} + \frac{(-1)^{8m+1}}{8m} = \frac{-1}{4m} = \frac{(-1)^{4m+1}}{4m}$

§3 More examples:

① $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt[3]{n}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/6}}$ is conditionally convergent

- $\sum_{n=1}^{\infty} \frac{1}{n^{1/6}}$ diverges (p-series with $p = \frac{1}{6} < 1$)
- $a_n = \frac{1}{n^{1/6}}$ is positive, $\lim_{n\to\infty} a_n \to 0$ & a_n is decreasing

($f(x) = \frac{1}{x^{1/6}}$ so $f'(x) = \frac{-1}{6}\frac{1}{x^{7/6}} < 0$ for $x > 0$)

So the series converges by the Alternating Series Test

② $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin^2 n}{n^{9/2}}$ is absolutely convergent:

- $\left| \frac{\sin^2 n}{n^{9/2}} \right| \leq \frac{1}{n^{9/2}}$ & $\sum_{n=1}^{\infty} \frac{1}{n^{9/2}}$ converges (p-series with $p = \frac{9}{2} > 1$)

So $\sum_{n=1}^{\infty} \left| \frac{\sin^2 n}{n^{9/2}} \right|$ converges by Comparison Test

③ $\sum_{n=1}^{\infty} (-1)^{n+1} \ln(\sqrt[n]{n}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$

- This series converges by the Alternating Series Test

(1) $a_n = \frac{\ln n}{n} > 0$ for $n \geq 2$; (2) $a_n \xrightarrow[n\to\infty]{} 0$ (by L'Hôpital)

(3) $f(x) = \frac{\ln x}{x}$ satisfies $f'(x) = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0$ for $x > 1$

So $a_n = f(n)$ is a decreasing sequence.

Q: Is it absolutely convergent?

Ratio Test? $\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{n+1} \Big/ \frac{\ln n}{n} = \frac{\ln(n+1)}{\ln(n)} \frac{n}{n+1} \longrightarrow 1$ inconclusive!

$$\left(\lim_{x\to\infty} \frac{f(x+1)}{f(x)} \underset{\substack{\downarrow \\ \frac{\infty}{\infty}\ \text{L'Hôp}}}{=} \lim_{x\to\infty} \frac{f'(x+1)}{f'(x)} = \lim_{x\to\infty} \underbrace{\boxed{\left(\frac{x}{x+1}\right)^2}}_{\downarrow x\to\infty\ 1} \frac{1 - \ln(x+1)}{1 - \ln x} = \right.$$

$$\lim_{x\to\infty} \frac{1-\ln(x+1)}{1-\ln x} \underset{\text{L'Hôp}}{=} \lim_{x\to\infty} \frac{-\frac{1}{x+1}}{-1/x} = \lim_{x\to\infty} \frac{x}{x+1} = 1 \quad)$$

$\frac{\infty}{\infty}$

ANSWER It is NOT absolutely convergent.

$$\left|(-1)^{n+1}\frac{\ln n}{n}\right| = \frac{\ln n}{n} > \frac{1}{n} \quad \text{for } n \geq 3 \quad \& \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

So $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ also diverges by Comparison Test.