Lecture LIV: Appendix A13: Absolute vs. Conditional Convergence

Recall: • A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Prop: An absolutely convergent series converges.

• The series is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

EXAMPLES: ① $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent

② $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent (its sum is $\ln 2$)

Proposition: Rearranging a series with positive terms does not change its sum.

Lecture 53: Rearranging $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ can change the sum.

THEOREM 1: An absolutely convergent series with sum S will have the same sum for any rearrangement.

Why? To show this, we introduce 2 positive sequences associated to $\sum_{n=1}^{\infty} a_n$:

$$\boxed{p_n := \frac{|a_n| + a_n}{2}} = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{else} \end{cases} \quad ; \quad \boxed{q_n = \frac{|a_n| - a_n}{2}} = \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{else} \end{cases}$$

Example: $a_n = -1, 2, 3, 0, 4, -10, -5, \ldots$

$p_n = 0, 2, 3, 0, 4, 0, 0, \ldots$

$q_n = 1, 0, 0, 0, 0, 10, 5, \ldots$

The type of convergence of a_n decides the convergence / divergence of $\sum_{n=1}^{\infty} p_n$ & $\sum_{n=1}^{\infty} q_n$

Lemma: (1) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then both $\sum_{n=1}^{\infty} p_n$ & $\sum_{n=1}^{\infty} q_n$ converge & furthermore $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$.

(2) If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then $\sum_{n=1}^{\infty} p_n$ & $\sum_{n=1}^{\infty} q_n$ diverge.

Proof: By construction, $a_n = p_n - q_n$ & $|a_n| = p_n + q_n$

(1) We know: convergent series can be added / substracted term-by-term.

Since $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} |a_n|$ converge then:

$$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \frac{|a_n| + a_n}{2} = \frac{1}{2}\left(\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} a_n\right) \text{ converges.}$$

$$\sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} \frac{|a_n| - a_n}{2} = \frac{1}{2}\left(\sum_{n=1}^{\infty} |a_n| - \sum_{n=1}^{\infty} a_n\right) \text{ converges.}$$

Also $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - q_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$ (this is a difference of 2 convergent series)

(2) Since $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then:

- $$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \frac{|a_n| + a_n}{2} = \frac{1}{2}\underbrace{\sum_{n=1}^{\infty} |a_n|}_{=\infty} + \frac{1}{2}\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{converges}} \text{ diverges}$$

- $$\sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} \frac{|a_n| - a_n}{2} = \frac{1}{2}\underbrace{\sum_{n=1}^{\infty} |a_n|}_{=\infty} - \frac{1}{2}\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{converges}} \text{ diverges}$$

Proof of Theorem 1:

Assume $\sum_{n=1}^{\infty} |a_n|$ converges to S. We know any rearrangement of it also has sum S.

Call $\sum_{n=1}^{\infty} b_n$ the rearrangement of $\sum_{n=1}^{\infty} a_n$. Then, $\sum_{n=1}^{\infty} |b_n|$ is convergent because it's a rearrangement of $\sum_{n=1}^{\infty} |a_n|$. In particular, its sum is S.

Now, use $p_n = \dfrac{|a_n| + a_n}{2}$ & $s_n = \dfrac{|b_n| + b_n}{2}$

$q_n = \dfrac{|a_n| - a_n}{2}$ $r_n = \dfrac{|b_n| - b_n}{2}$

By construction, $\{s_n\}$ is a rearrangement of $\{p_n\}$ so $\sum_{n=1}^{\infty} s_n = \sum_{n=1}^{\infty} p_n$

$\{r_n\}$ ———————— $\{q_n\}$ so $\sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} q_n$

In particular, by Lemma (1) applied to $\{a_n\}$ & $\{b_n\}$ in (*), we have

$$\sum_{n=1}^{\infty} b_n \underset{(*)}{=} \sum_{n=1}^{\infty} (s_n - r_n) = \sum_{n=1}^{\infty} s_n - \sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n \underset{(*)}{=} \sum_{n=1}^{\infty} a_n$$

So the sum is preserved.

Q: What about conditionally convergent series?

THEOREM 2: Assume $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Then, its terms can be rearranged to give a convergent series with any prescribed sum s in $\mathbb{R}$, and also a divergent series with sum ∞ or $-\infty$.

Proof of Theorem 2 (due to Riemann)

1. First, we fix the desired value s we want to get. We know by the Lemma that $\sum_{n=1}^{\infty} p_n$ & $\sum_{n=1}^{\infty} q_n$ both diverge (sum $= \infty$)

- We start by writing down p's until the partial sum $p_1 + \cdots + p_{n_1}$ is s or exceeds s for the first time.

$$p_1 + p_2 + \cdots + p_{n_1} \geq s \qquad (\text{but } p_1 + \cdots + p_{n_1 - 1} < s)$$

- Then, we substract q's until the result is s or less for the first time

$$p_1 + \cdots + p_{n_1} - q_1 - q_2 - \cdots - q_{m_1} \leq s \qquad (\text{but } p_1 + \cdots + p_{n_1} - q_1 - \cdots - q_{m_1 - 1} > s)$$

- Then add p's (starting from p_{n_1+1}) until we get s or more for the first time

$$p_1 + \cdots + p_{n_1} - q_1 - \cdots - q_{m_1} + p_{n_1+1} + \cdots + p_{n_2} \geq s$$

$$(\text{but } p_1 + \cdots + p_{n_1} - q_1 - \cdots - q_{m_1} + p_{n_1+1} + \cdots + p_{n_2 - 1} < s)$$

- Next, we substract q's, starting from $m_1 + 1$ unless we get s or less, and so on.

Note that each step can be achieved because $\sum_{n\geq k} p_n = \infty$ & $\sum_{n\geq k} (-q_n) = \infty$ for any fixed $k \geq 0$.

EXAMPLE: $S = 3$

n =	1	2	3	4	5	6	7	8	9	
a_n =	-1,	2,	3,	0,	4,	-10,	5,	-1,	...	
p_n =	0,	2,	3,	0,	4,	0,	5,	0,	...	
q_n =	1,	0,	0,	0,	0,	10,	0,	1,	...	

- $p_1 + p_2 = 2 < 3$ & $p_1 + p_2 + p_3 = 5 > 3$ $\quad n_1 = 3$
- $p_1+p_2+p_3-q_1-q_2-q_3-q_4-q_5 = 4 > 3$ &
  $p_1+p_2+p_3-q_1-q_2-q_3-q_4-q_5-q_6 = -6 < 3$ $\quad m_1 = 6$
- $p_1+p_2+p_3-q_1-q_2-q_3-q_4-q_5-q_6+p_4+p_5+p_6+p_7 = -2 < 3$
  $p_1+p_2+p_3-q_1-q_2-q_3-q_4-q_5-q_6+p_4+p_5+p_6+p_7+p_8 = 3 \geq 3$ $\quad n_2 = 8$
- $p_1+p_2+p_3-q_1-q_2-q_3-q_4-q_5-q_6+p_4+p_5+p_6+p_7+p_8-q_7 = 3 \geq 3$
  $p_1+p_2+p_3-q_1-q_2-q_3-q_4-q_5-q_6+p_4+p_5+p_6+p_7+p_8-q_7-q_8 = 2 < 3$
  $m_2 = 8$

This partial sum starts a rearrangement for $\{a_n\}$'s (after ignoring 0's)

$$\begin{array}{cccccccccccccccc} p_1 & +p_2 & +p_3 & -q_1 & -q_2 & -q_3 & -q_4 & -q_5 & -q_6 & +p_4 & +p_5 & +p_6 & +p_7 & +p_8 & -q_7 & -q_8 \\ 0 & a_2 & a_3 & a_1 & 0 & 0 & 0 & 0 & a_6 & a_4 & a_5 & 0 & a_7 & 0 & 0 & a_8 \end{array}$$

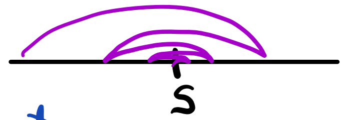
In general, we have:

Claim 1: The sums $p_1 + \cdots + p_{n_1} - q_1 - \cdots - q_{m_1} + p_{n_1+1} + \cdots + p_{n_2} - q_{m_1+1} - \cdots$ give a rearrangement of a_n's with 0's that can be ignored.

Reason: $p_n = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{else} \end{cases}$ & $-q_n = \begin{cases} 0 & \text{if } a_n \geq 0 \\ a_n & \text{if } a_n < 0 \end{cases}$

By construction each a_n appears as p_n or q_n & only once!

Claim 2 The rearrangement has sum S because $a_n \xrightarrow[n\to\infty]{} 0$, $p_n \xrightarrow[n\to\infty]{} 0$ & $q_n \xrightarrow[n\to\infty]{} 0$ & by construction $n_1, m_1, n_2, m_2, \dots$ are chosen so that the partial sums get closer to S as we move along.



Indeed: given $\varepsilon > 0$ we can find m_0 & $n_0 > 0$ so that

$$p_n = |p_n| < \varepsilon \quad \text{for } n \geq n_0$$

&

$$q_m = |q_m| < \varepsilon \quad \text{for } m \geq m_0$$

Take $\boxed{T_r} := \sum_{i=1}^{n_1} p_i - \sum_{i=1}^{m_1} q_i + \sum_{i=n_1+1}^{n_2} p_i - \sum_{i=m_1+1}^{m_2} + \dots + \sum_{i=n_{r-1}+1}^{n_r} p_i$

$\boxed{U_r} := \sum_{i=1}^{n_1} p_i - \sum_{i=1}^{m_1} q_i + \sum_{i=n_1+1}^{n_2} p_i - \sum_{i=m_1+1}^{m_2} + \dots + \sum_{i=n_{r-1}+1}^{n_r} p_i - \sum_{i=m_{r-1}+1}^{m_r} q_i$ for $r \geq 1$

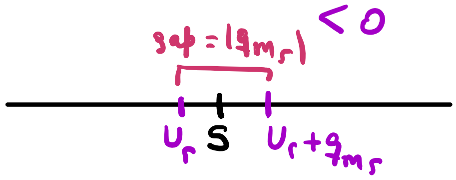
Then $U_r \leq S$ $\quad T_r \geq S$ for all r

Pick r so that $n_r \geq n_0$ & $m_r \geq m_0$

Then $|U_r - S| = S - U_r = \underbrace{S - (T_r - q_{m_{r-1}+1} - \dots - q_{m_r-1})}_{<0} - q_{m_r} > 0$

$\quad (T_r - q_{m_{r-1}+1} - \dots - q_{m_r-1}) = U_r + q_r$

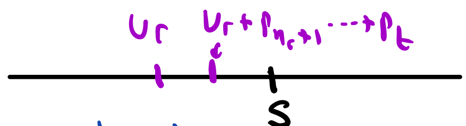
forcing $|U_r - S| \leq |q_{m_r}|$



Next we compute: $|U_r + p_{n_r+1} + \dots + p_t - S|$ for $n_r + 1 \leq t \leq n_r$

- If $t < n_r$:

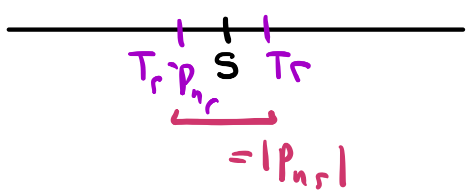
$U_r \leq U_r + p_{n_r+1} + \dots + p_t < S$



so $|(U_r + p_{n_r+1} + \dots + p_t) - S| < S - U_r < |q_{n_r}|$

- If $t = n_r$, then $U_r + p_{n_r+1} + \dots + p_{n_r} = T_r > S$ &

$T_r - p_{n_r} = U_r + p_{n_r+1} + \dots + p_{n_r-1} < S$



So $|T_r - S| \leq |p_{n_r}|$

- Similarly $|T_r - q_{m_r+1} - \dots - q_t - S| \leq S - T_r$ for all $m_r + 1 \leq t < m_{r+1}$

& $U_{r+1} = T_r - q_{m_r+1} - \cdots - q_{m_{r+1}}$ satisfies $|U_{r+1} - s| \leq |q_{m_{r+1}}|$

Since $|q_{m_r}|, |q_{m_{r+1}}|, |p_{n_r}| < \varepsilon$ we conclude that

$|T_r - s|$, $|U_r - s|$ & all partial sums of ps & $-q$s in between T_r & U_r differ from s in less than ε.

2 To make $\sum b_n = +\infty$ we take n_1 so that $p_1 + \cdots + p_{n_1} \geq 1$ for the first time. Then take $p_1 + \cdots + p_{n_1} - q_1$ & add enough ps so that

$p_1 + \cdots + p_{n_1} - q_1 + p_{n_1+1} + \cdots + p_{n_2} \geq 2$ for the first time.

Then substract q_2 & add ps to be ≥ 3, etc.

By construction, this series rearranges $\sum a_n$ (same reason as claim 1) & since $p_n, q_n \xrightarrow[n\to\infty]{} 0$ we get that the partial sums go to $+\infty$.

3 To make $\sum b_n = -\infty$, reverse the roles of p & q in 2. That is pick m_1 so that $-q_1 - q_2 - \cdots - q_{m_1} \leq -1$ for the first time, then add p_1 & substract enough q's so that we get

$-q_1 - q_2 - \cdots - q_{m_1} + p_1 - q_{m_1+1} - \cdots - q_{m_2} \leq -2$, etc.