Lecture LV : §14.1 Introduction to power series
§14.2 The interval of convergence of a power series

GOALS: ① Given a power series $\sum_{n=1}^{\infty} a_n x^n$ (convergent for $x=0$), determine all values of x that make it convergent / absolutely convergent / conditionally convergent.

② Try to recognize this power series (viewed as a function of x) as a known function (elementary, whenever possible). This is often used to recognize solutions to differential equations.

§1. Examples:

① Fix $f(x) = \sum_{n=0}^{\infty} x^n$ (geometric series)

- We know $f(x)$ converges absolutely for $0<|x|<1$ by Ratio Test

$a_n = |x^n|$ so $\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}}{|x|^n} = |x|$

If $|x|<1$: $\sum_{n=1}^{\infty} a_n$ converges

If $|x|>1$: $\sum_{n=1}^{\infty} a_n$ diverges

- $f(0) = 1$ so it converges
- If $x>1$ $x^n \longrightarrow \infty \neq 0$ so $\sum_{n=0}^{\infty} x^n$ diverges
- If $x=1$ $x^n = 1 \longrightarrow 1 \neq 0$ so $\sum_{n=0} 1^n$ ______
- If $x=-1$ $x^n = (-1)^n$ has no limit so $\sum_{n=0}^{\infty} (-1)^n$ diverges
- If $x<-1$ x^n ____________ so $\sum_{n=0}^{\infty} x^n$ diverges

Conclusion: Domain of $f = (-1,1)$ & $f(x)$ converges absolutely in its domain.

Furthermore, $f(x) = \frac{1}{1-x}$ for $-1<x<1$. (*)

Note that $\frac{1}{1-x}$ cannot be evaluated at $x=1$. In particular, the equality in (*) is only valid for $-1<x<1$ & cannot be extended beyond this interval.

LSSE)

(2) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$ for all x.

Again: we use the Ratio Test to check the series converges absolutely for all values of $x \neq 0$ (for $x=0$ we get 1)

$$a_n = \frac{|x|^n}{n!} \quad \text{so} \quad \frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}}{(n+1)!} \Big/ \frac{|x|^n}{n!} = \frac{|x|^{n+1}}{|x|^n} \frac{n!}{(n+1)!} = \frac{|x| \; n!}{(n+1)n!} = \frac{|x|}{n+1}$$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$ so the series converges absolutely for all $x \neq 0$.

Q: Why is the series $= e^x$?

For this, we need to remember that e^x is the unique solution to the differential equation $\begin{cases} f'(x) = f(x) \\ f(0) = 1 \end{cases}$ (initial condition)

We next confirm that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ also solves the same equation & with the same initial condition. This forces both functions to agree!

Q How to solve $f' = f$ via a power series?

Propose $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ & differentiate term-by-term

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Identity of series is guaranteed by term-by-term equality:

constant: $a_0 = a_1$

x-term: $a_1 = 2a_2 \rightsquigarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2}$

x^2-term: $a_2 = 3a_3 \rightsquigarrow a_3 = \frac{a_2}{3} = \frac{a_0}{3.2} = \frac{a_0}{3!}$

x^3-term: $a_3 = 4a_4 \rightsquigarrow a_4 = \frac{a_3}{4} = \frac{a_0}{4.3.2} = \frac{a_0}{4!}$

In general: $a_n = \frac{a_0}{n!}$

So $f(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0} \frac{x^n}{n!}$ (convergent everywhere)

If $f(0) = 1$, we get $a_0 = 1$.

③ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$

Q: Where is this identity valid?

- We know $\ln(1+x)$ is only valid for $1+x>0$, so $x>-1$.
- The series converges absolutely for $0<|x|<1$ by the Ratio Test:

$$a_n = \left|\frac{(-1)^{n+1}x^n}{n}\right| = \frac{|x|^n}{n} \qquad \frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}}{n+1} \Big/ \frac{|x|^n}{n} = |x|\,\frac{n}{n+1} \xrightarrow[n\to\infty]{} |x|$$

- So it converges for $|x|<1$
- Does not converge absolutely for $|x|>1$.
- If $|x|>1$, then $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^n}{n}\right| = \infty \neq 0$ so the series diverges

because $\lim_{n\to\infty} \frac{(-1)^{n+1}x^n}{n} \neq 0$.

$$\Big(\text{Indeed, } \underbrace{\lim_{n\to\infty}\frac{|x|^n}{n}}_{\frac{\infty}{\infty}} = \lim_{t\to\infty}\frac{e^{t\ln|x|}}{t} \underset{\substack{\text{L'Hôpital} \\ (t \text{ is the variable})}}{=} \lim_{t\to\infty}\frac{\ln|x|\,e^{t\ln|x|}}{1} = \lim_{t\to\infty}\frac{|x|^t\ln|x|}{1} = \infty\Big)$$

- If $x=1$: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}1^n}{n}$ converges (only conditionally) by the Alternating series Test ($b_n = \frac{1}{n} > 0$, $b_n \xrightarrow[n\to\infty]{} 0$ & (b_n) is decreasing)

We saw in Lecture 52 that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln(2) = \ln(1+1)$.

- If $x=-1$, then $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n} = -1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots$

diverges (this is $-\sum_{n=1}^{\infty}\frac{1}{n}$ & we know the harmonic series diverges).

Conclusion: The power series converges for $(-1,1]$

Q: How to guess that $\ln(1+x)$ agrees with this power series in $(-1,1]$?

A: Start with the identity $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n$ valid on $(-1,1)$

Integrate the series term-by-term to get

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int -x + x^2 - x^3 + x^4 - \cdots \, dx$$

$$= -\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots + C = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} + C$$

To determine C we evaluate at $x=0$:

$$0 = \ln(1) = 0 + C \quad \text{so} \quad C = 0.$$

We conclude: $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ in $(-1,1)$ & also valid for $x=1$ by a separate calculation.

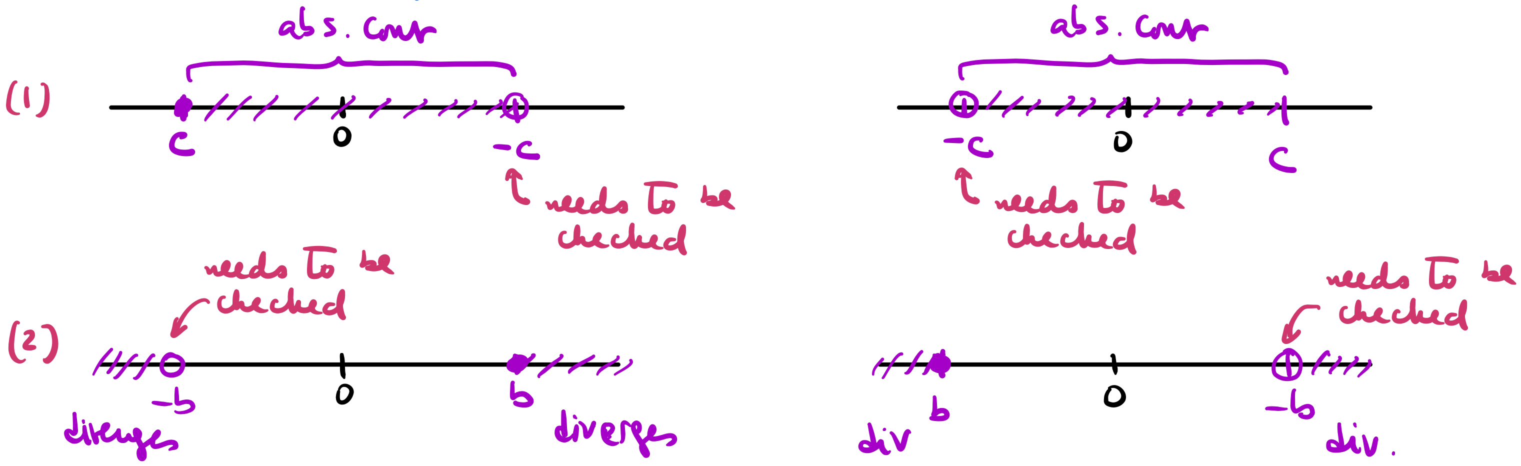
These 3 examples show us that given a power series $\sum_{n=0}^{\infty} a_n x^n$ we first must determine the region where it converges & only then we can try to recognize this function as an elementary one. This function may not be defined everywhere & its domain can be larger than the domain of the power series.

Q To what extent are the examples above typical?

Useful Lemma: (1) If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for some $x=c$ & $c \neq 0$ then it converges absolutely for all x with $|x| < |c|$.

(2) If the power series $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x=b$, then it diverges for all x with $|x| > |b|$.

We'll discuss the proof in the next lecture. Pictorially:



We check the statement in the examples

① $\sum_{n=0}^{\infty} x^n$ diverges for $x=1$ so diverge for all $|x|>1$

- The case $x=-1$ needs to be addressed separately: it diverges.
- For each $0<c<1$ the series converges at $x=c$ so it converges absolutely for $|x|<c$ (this works for all c, eg $c=\frac{1}{2}$)

② $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for each $x=c\neq 0$ so it converges absolutely in $|x|<|c|$

③ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ converges for $x=1$, so it converges absolutely for $|x|<1$.

- It diverges for $x=-1$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$, so it diverges for $|x|>1$.

Conclusion:
- Absolute convergence for $-1<x<1$
- Conditional convergence for $x=1$
- Divergence for $x\leq -1$ & $x>1$