Lecture LVI: §14.2 The interval of convergence of a power series

Recall: $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$ is a power series in x

- $f(x)$ is a function of x, $f(0) = a_0$

Q1: What's the domain of f? **Name**: Interval of Convergence.

Q2: Can we find an elementary function representing f? **Tools**: Integrate or Differentiate Term-by-Term, view $f(x)$ as a solution to a differential equation with elementary solution (**Example**: e^x solves $\begin{matrix} f' = f \\ f(0) = 1 \end{matrix}$ & so does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$)

Q3: Can we find power series representing any function? $\rightsquigarrow$ Taylor series.

EXAMPLE: $f(x) = \sum_{n=0}^{\infty} x^n$ is defined for $-1 < x < 1$ & it equals $\frac{1}{1-x}$ in this range.

§1 Interval of Convergence:

Very Useful Theorem: Consider $f(x) = \sum_{n=0}^{\infty} a_n x^n$:

① If $f(x)$ converges at $x = c$ & $c \neq 0$, then it converges absolutely for all x with $|x| < |c|$

② If $f(x)$ diverges at $x = b$, then it diverges for all x with $|x| > |b|$

Example above: $f(x)$ diverges at $x = 1$ so it diverges for $|x| > 1$

$f(x)$ converges for any $|c| < 1$ so it converges absolutely for $|x| < c$ $\rightsquigarrow$ converges absolutely in $(-1, 1)$.

Only missing pt is $x = -1$, We check by hand it diverges.

EXAMPLE 2: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for any x. $(= e^x)$

Why? Fix any $c > 0$.

Ratio Test $\frac{b_{n+1}}{b_n} = \frac{c^{n+1}}{(n+1)!} \Big/ \frac{c^n}{n!} = \frac{c}{n+1} \xrightarrow[n \to \infty]{} 0 < 1$ so converges for any $c > 0$

So By the Theorem, we have absolute convergence in $(-c, c)$. But $c>0$ is arbitrary, so we have absolute convergence everywhere.

Consequence: We have 3 possible scenarios:

1. Absolute convergence everywhere
2. Convergence only at $x=0$
3. We can find $r>0$ minimal where f is abs. convergent in $(-r, r)$ & divergent for $|x|>r$ (r = radius of convergence = ROC). The cases $x=r$ & $x=-r$ have to be checked by hand separately.

Proof of the Theorem: We use comparison theorems.

1. Suppose f converges for $x=c$ & $c\neq 0$, then $a_n c^n \xrightarrow[n\to\infty]{} 0$. This means that eventually (for $n \geq n_0$) we must have $|a_n c^n| < 1$ (Take $\varepsilon=1$ in the definition of limit)

But then if $|x|<|c|$, we have

$$|a_n x^n| = |a_n (\tfrac{x}{c} c)^n| = \underbrace{|a_n c^n|}_{<1} \left|\frac{x}{c}\right|^n < \left|\frac{x}{c}\right|^n \quad \text{for } n \geq n_0.$$

Setting $r = \left|\frac{x}{c}\right| < 1$ gives

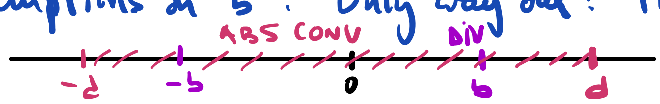
$$\sum_{n=n_0} |a_n x^n| \leq \sum_{n=n_0}^{\infty} r^n = r^{n_0} + r^{n_0+1} + \cdots = r^{n_0} \sum_{n=0} r^n = \frac{r^{n_0}}{1-r}$$

$0<r<1$

By Comparison $\sum_{n=n_0}^{\infty} |a_n x^n|$ converges, so $\sum_{n=0}^{\infty} |a_n x^n|$ converges as well.

Conclusion: $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<|c|$.

2. We argue by contradiction. Suppose $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x=b$ but converges for $x=d$ with $|d|>|b|$. Then by 1 the series converges absolutely for $x=b$ since $|b|<|d|$. But this cannot happen by our assumptions on b. Only way out: The series diverges for any x with $|x|>|b|$



Consequence: Given $\sum_{n=0}^{\infty} a_n x^n$ precisely ONLY ONE of the following 3 situations occurs:

1. The series converges for all x (and absolutely!) $[R=+\infty]$
2. ——— only for $x=0$ $[R=0]$
3. There is a positive real number R (= radius of convergence) for which the series • converges absolutely for $|x|<R$ & • diverges for $|x|>R$ $[0<R<\infty]$

⚠ Nothing can be said a priori about $x=\pm R$. In each example, we must treat these 2 values separately

Intervals of convergence:
1. $\mathbb{R}=(-\infty,\infty)$
2. $[0,0]$
3. $(-R,R)$, $(-R,R]$, $[-R,R)$ or $[-R,R]$

§2. Examples:

In general: R = ROC is computed with Ratio or Root Test.

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad R=+\infty$ by Ratio Test.
2. $\sum_{n=0}^{\infty} n!\, x^n$ $\quad R=0$

Ratio Test ($x\neq 0$): $\frac{a_{n+1}}{a_n}=\frac{(n+1)!\;|x|^{n+1}}{n!\;\;|x|^n}=|x|(n+1)\xrightarrow[n\to\infty]{}\infty$ for all $x\neq 0$.

So no absolute convergence for any $x\neq 0$.

3. (a) $(-R,R)$ $\quad \sum_{n=0}^{\infty} x^n$ $\quad (R=1)$

(b) $(-R,R]$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$ $\quad (R=1)$

(c) $[-R,R)$ $\quad \sum_{n=0}^{\infty} \frac{x^n}{n+1}$ $\quad (R=1)$

Compute ROC using Ratio Test:

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{|x|^{n+1}}{n+1} \Big/ \frac{|x|^n}{n} = |x| \frac{n}{n+1} \longrightarrow |x|$$

- If $|x|<1$ we converge (absolutely)
- If $|x|>1$ we diverge

$\Big\}$ This forces $R=1$.

Endpoint analysis:

- $\underline{x=1}$: $\displaystyle\sum_{n=0}^{\infty} \frac{1^n}{n+1} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges
- $\underline{x=-1}$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \ln 2$ (converges by

Alternating Series Test $a_n = \frac{1}{n+1} > 0$, $a_n \to 0$ & a_n decreasing)

(d) $[-R, R]$ $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ $(R=1)$

Compute ROC by Ratio Test:

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{|x|^{n+1}}{(n+1)^2} \Big/ \frac{|x|^n}{n^2} = |x|\left(\frac{n}{n+1}\right)^2 \longrightarrow |x|$$

- If $|x|<1$ we converge (absolutely)
- If $|x|>1$ we diverge

$\Big\}$ This forces $R=1$.

Endpoints: $\underline{x=1}$ $\displaystyle\sum_{n=0}^{\infty}\frac{1^n}{n^2} = \sum_{n=0}^{\infty}\frac{1}{n^2}$ p-series with $p=2>1$, so it converges!

$\underline{x=-1}$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}$ is absolutely convergent, so it converges.

Q What if we have $\displaystyle\sum_{n=0}^{\infty} a_n \underbrace{(x-c)}_{=z}{}^n$?

1. Compute R = radius of convergence for $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ ⤳ ROC for $f(x)$ is also R
2. Interval of convergence for $\displaystyle\sum_{n=0}^{\infty} a_n z^n$, say $[-R, R)$, then IOC for $f(x)$ is $[-R+c, R+c)$ (shift IOC by c).