Lecture LVII: §14.3 Differentiation & Integration of power series
§14.4 Taylor series & Taylor's formula

§1 Derivatives & Integrals of power series:

Thinking of power series as polynomials with infinite tails, we ask:

Q: Can we differentiate / integrate power series term-by-term?

A: YES, but the reasons are rather delicate (we need "uniform convergence" from Appendix A15)

Theorem: Fix $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$, so $f(x)$ is a function on $(-R, R)$. Then:

(1) $f(x)$ is continuous on $(-R,R)$

(2) $f(x)$ is differentiable on $(-R,R)$ & we get $f'(x)$ by term-by-term differentiation:
$$f'(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)' = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)'$$
$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

(3) $f(x)$ can be integrated term-by-term on $(-R,R)$:
$$\int_0^x f(t)\,dt = \int_0^x \sum_{n=0}^{\infty} a_n t^n\,dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt = \sum_{n=0}^{\infty} a_n \frac{t^{n+1}}{n+1}\Big|_{t=0}^{t=x} = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$$

Observe: We can repeat the process in (2) (f' is a power series & its ROC is $= R$) and get that $f(x)$ is infinitely differentiable on $(-R,R)$ &
$$\frac{d^k f}{dx^k} = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n\, x^{n-k} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \quad \text{for } k=1,2,\ldots$$

§2. Application 1: find elementary functions for power series

EXAMPLE 1: $f(x) = \dfrac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$ for x in $(-1,1)$.

Integrate term-by-term to get
$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x \sum_{n=0}^{\infty} (-t)^n\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^{n+1}\Big|_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This equality is valid on $(-1,1)$

Q Can we push it to $x = \pm 1$?

A For $x=1$ $\quad \ln(2) \overset{?}{=} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ True ✓

For $x=-1$ $\quad \underset{-\infty}{\underset{\|}{\ln(0)}} \overset{?}{=} -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \cdots = -\sum_{n=1}^{\infty} \frac{1}{n}$ both diverge.

EXAMPLE 2: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ has $R = +\infty$ by Ratio Test

Compute f' term-by-term:

$$f' = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{n\,x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = f.$$

So f is a solution to the differential equation $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$. By uniqueness of solutions, we get $f(x) = e^x$ (for all x in $\mathbb{R}$)

EXAMPLE 3: $\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$

Claim: ROC $= 1$.

Why? Use Ratio Test!

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{(n+1)^2 |x|^{n+1}}{n^2 |x|^n} = \left(\frac{n+1}{n}\right)^2 |x| \xrightarrow[n\to\infty]{} |x|$$

$\left.\begin{array}{l} \text{Converges if } |x|<1 \\ \text{Diverges if } |x|>1 \end{array}\right\}$ so by definition $R=1$.

So $f(x) = \sum_{n=1}^{\infty} n^2 x^n = x\,\underbrace{(1 + 2^2 x + 3^2 x + 4^2 x^3 + \cdots)}_{=:\,g(x)}$ is defined on $(-1,1)$. It diverges for $x = \pm 1$ since $n^2(\pm 1)^n \underset{n\to\infty}{\not\to} 0$.

In particular $g(x)$ also has ROC $= 1$.

Q: What function is represented by $f(x)$?

A: We work with g instead. $g(x) = \sum_{n=0}^{\infty} (n+1)^2 x^n$

Integrate $g(x)$ term-by-term $h(x) = \int_0^x g(t)\,dt$ also has ROC$=1$.

$$h(x) = \sum_{n=0}^{\infty} \int_0^x (n+1)^2 t^n \, dt = \sum_{n=0}^{\infty} (n+1)^2 \frac{t^{n+1}}{n+1} \Big|_0^x = \sum_{n=0}^{\infty} (n+1) x^{n+1}$$

$$= x + 2x^2 + 3x^3 + \cdots = x \underbrace{(1 + 2x + 3x^2 + \cdots)}_{=: j(x)}$$

- The function j also has ROC $= 1$. $\quad j(x) = \sum_{n=0}^{\infty} (n+1) x^n$

We integrate j term-by-term & get a new function $P(x)$ with ROC $=1$

$$P(x) = \int_0^x j(t)\,dt = \int_0^x \sum_{n=0}^{\infty} (n+1) t^n \, dt = \sum_{n=0}^{\infty} \int_0^x (n+1) t^n \, dt = \sum_{n=0}^{\infty} t^{n+1} \Big|_0^x$$

$$= \sum_{n=0}^{\infty} x^{n+1} = x + x^2 + x^3 + \cdots = x(1 + x + x^2 + \cdots) \underset{-1<x<1}{=} \frac{x}{1-x}$$

Now we reverse the process by the Fundamental Theorem of Calculus:

- $P(x) = \frac{x}{1-x} \quad \rightsquigarrow \quad \left(\frac{x}{1-x}\right)' = P(x)' = j(x)$

$$j(x) = \frac{(1-x) - x(-1)}{(1-x)^2} = \frac{1}{1-x^2}$$

- $h(x) = x\, j(x) = \frac{x}{(1-x)^2}$

So $g(x) = h'(x) = \frac{1(1-x)^2 - x\, 2(1-x)(-1)}{(1-x)^4} = \frac{(1-x)(1-x+2x)}{(1-x)^3} = \frac{1+x}{(1-x)^3}$

- $f(x) = x\, g(x) = \frac{x(1+x)}{(1-x)^3}$

Conclusion: $f(x) = \frac{x(1+x)}{(1-x)^3}$ can be expressed as a power series on $(-1,1)$, namely $\sum_{n=1}^{\infty} n^2 x^n$

§3 Application 2: Taylor series

Fix $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with ROC $= R > 0$.

- We know $f, f', f'', \ldots, f^{(k)}, \ldots$ all exist & are power series with ROC $= R > 0$

$$f'(x) = \boxed{a_1} + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$f''(x) = \boxed{2a_2} + 3!\, a_3 x + 4 \cdot 3\, a_4 x^2 + \cdots$$

$f'''(x) = \boxed{3!\,a_3} + 4\cdot 3\cdot 2\, a_4 x + \cdots$

$f^{(4)}(x) = \boxed{4!\,a_4} + \cdots$

• In general: $f^{(n)}(x) = n!\,a_n +$ terms containing x as a factor (series vanishing at $x=0$ with ROC $= R$)

So $f^{(n)}(0) = n!\,a_n + 0 = n!\,a_n$ gives

TAYLOR COEFFICIENT: $a_n = \dfrac{f^{(n)}(0)}{n!}$ for all $n = 0, 1, 2, \ldots$ ($0! = 1$ by convention)

Conclusion: If a function can be represented by a power series with ROC $= R > 0$ then, the series must be the Taylor series (around $x=0$), i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Q: Can we reverse this? Meaning, assuming we have a function f which is differentiable up to any order at $x=0$, we can write down its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Does this series represent $f(x)$?

A: NOT always!

EXAMPLE: (Ex 42 §12.3) $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ We can compute all $f^{(n)}(0)$ by definition & we get $f^{(n)}(0) = 0$ for all n. This means that the Taylor series is $\sum_{n=0}^{\infty} 0\, x^n = 0$. But f is not the constant function 0, it's just almost flat near 0.

Easy extension: Taylor series of a function f around a fixed point $x=c$ in the domain of f is $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$.

Natural question: When can we represent f by its Taylor series around a point c in its domain? ie when is $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$? $(*)$

It's enough to focus on the case $c=0$. Assume the Taylor series has ROC $= R > 0$. We can decide (*) by estimating the error of approximating f by a truncation of the Taylor series. This is the so called <u>Lagrange Remainder Formula</u>.

$$S_N(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(N)}(0)}{N!}x^N$$

polynomial of degree $\leq N$ in x

Call $R_N^{(f)}(x) = S_N(x) - f(x)$ (Remainder)

<u>Proposition</u>: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ converges to $f(x)$ if, and only if, $|R_N^{(f)}(x)| \xrightarrow[N\to\infty]{} 0$ for every x in $(-R, R)$

⚠ This becomes a useful statement only if we have a formula for $R_N(f)(x)$ that we can use to check $|R_N(f)(x)| \xrightarrow[N\to\infty]{} 0$

<u>Lagrange Remainder Formula</u>

$$R_N(f)(x) = \frac{f^{(N+1)}(b)}{(N+1)!}x^{N+1} \quad \text{for some } b \text{ between } 0 \text{ \& } x.$$

<u>Obs</u>: We can extend this for Taylor series expansions around c. Then:

$$R_N(f)(x) = \frac{f^{N+1}(b)}{(N+1)!}(x-c)^{N+1} \quad \text{for some } b \text{ between } c \text{ \& } x.$$