Lecture LVIII: §14.4 (cont.) Taylor series & Taylor's formula

§1 Taylor series:

Recall: The Taylor series of a function f centered at c (with c in the domain of f) is $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

We need f to be differentiable up to any order at $x=c$.

Q: When is f equal to its Taylor series around c?

A NOT always (example for last lecture)

Next approach: Truncate the Taylor series by working with a partial sum & estimate the error in the approximation.

Write $S_N(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(N)}(c)}{N!}(x-c)^N$

This is the Taylor polynomial of degree $\leq N$ ($=N$ if $f^N(c) \neq 0$)

Write $\boxed{R_N(f)(x) = f(x) - S_N(x)}$

Proposition: $\sum_{N=0}^{\infty} \frac{f^{(N)}(c)}{N!}(x-c)^N$ with ROC $= R > 0$ converges to $f(x)$ if, and only if $|R_N(f)(x)| \xrightarrow[N\to\infty]{} 0$ for every x in $(-R+c, R+c)$.

⚠ . The N_0 that works for one x may not be large enough for another x.

. This result is ONLY useful for practical purposes if we have a formula for $R_N(f)(x)$

Lagrange Remainder Formula: $R_N(f)(x) = \frac{f^{(N+1)}(b)}{(N+1)!}(x-c)^{N+1}$ for b in between x & c.

Q: Why should we expect this?

. $N=0$: $R_0(f)(x) = f(x) - f(c) = f'(b)(x-c)$ by the Mean Value Thm

. The proof for higher N will involve the Mean Value Thm as well.

Another reason: Say $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ with ROC $= R > 0$

Then: $$f(x) - S_N(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n - \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n$$
$$= \sum_{n=N+1}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = \frac{f^{(N+1)}(c)}{(N+1)!}(x-c)^{N+1} + \cdots$$

Lagrange Formula replaces this tail by $\frac{f^{(N+1)}(b)}{(N+1)!}(x-c)^{N+1}$ & this value looks very similar to the 1st term of the tail.

§2. Applications:

① Show $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

- We know by the Ratio Test that ROC $= R > 0$.

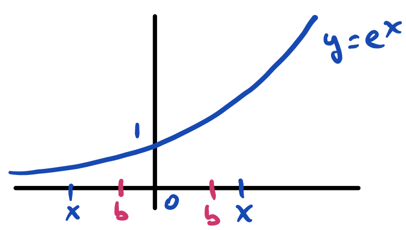
Q: What is the Taylor series of e^x around $x=0$?

$$\left.\begin{array}{lcl} f(x) = e^x & \rightsquigarrow & f(0) = e^0 = 1 \\ f'(x) = e^x & \rightsquigarrow & f'(0) = 1 \\ \vdots & & \\ f^{(n)}(x) = e^x & \rightsquigarrow & f^{(n)}(0) = 1 \end{array}\right\}$$ so the Taylor series is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Remainder? $$|R_N(e^x)(x)| = \left| \frac{f^{(N+1)}(b)}{(N+1)!} x^{N+1} \right| = \left| \frac{e^b}{(N+1)!} x^{N+1} \right| = \frac{|x|^{N+1} |e^b|}{(N+1)!}$$

with b in between 0 & x.

- If $x > 0$ $\quad |e^b| = e^b < e^x$
- If $x < 0$ $\quad |e^b| = e^b < e^0 = 1$



$M = \max\{e^x, 1\}$

So we can bound $|R_N(e^x)(x)|$ independently of b! We get:

$$|R_N(e^x)(x)| \leq \frac{|x|^{N+1}}{(N+1)!} M \xrightarrow[N\to\infty]{} 0 \cdot M = 0 \quad \left(\text{because } \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ converges}\right).$$

<u>In conclusion</u>: $R_N(e^x)(x) \xrightarrow[N\to\infty]{} 0$ for every <u>fixed</u> x & so e^x is represented by its Taylor series, as we already knew.

② Taylor series for $\sin(x)$: We take $x=0$ as the center.

$f_{(x)} = \sin x \rightsquigarrow f_{(0)} = 0$

$f'_{(x)} = \cos x \rightsquigarrow f'_{(0)} = 1$

$f''_{(x)} = -\sin x \rightsquigarrow f''_{(0)} = 0$

$f^{(3)}(x) = -\cos x \rightsquigarrow f^{(3)}_{(0)} = -1$

$f^{(4)}_{(x)} = \sin x \rightsquigarrow f^{(4)}_{(0)} = 0$ & we repeat from here on.

Conclude: Taylor series only involves ODD powers of x & it alternate signs $+ - + -$, etc

$$\text{Taylor series} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \overbrace{(-1)^n \frac{x^{2n+1}}{(2n+1)!}}^{=: a_n}$$

Only odd powers is consistent with $\sin x$ being an ODD function ($\sin(-x) = -\sin x$)

• Radius of convergence = ? Use Ratio Test:

$$b_n = |a_n| = \frac{|x|^{2n+1}}{(2n+1)!}$$

$(2n+3)! = (2n+3)(2n+2)(2n+1)!$

$$\frac{b_{n+1}}{b_n} = \frac{|x|^{2n+3}}{(2n+3)!} \Big/ \frac{|x|^{2n+1}}{(2n+1)!} = \frac{|x|^2 (2n+1)!}{(2n+3)!} = \frac{|x|^2}{(2n+3)(2n+2)} \xrightarrow[n\to\infty]{} 0$$

for any fixed x

So RoC $= +\infty$.

• Remainder formula:

$$|R_N(\sin x)_{(x)}| = \left| \frac{f^{(N+1)}_{(b)}}{(N+1)!} x^{N+1} \right| = \frac{|x|^{N+1}}{(N+1)!} |f^{(N+1)}_{(b)}|$$

& $f^{(N+1)}_{(b)} = \pm \sin b$ or $\pm \cos b$ In both cases $|f^{(N+1)}_{(b)}| \leq 1$

So $|R_N(\sin x)_{(x)}| \leq \dfrac{|x|^{N+1}}{(N+1)!} \xrightarrow[N\to\infty]{} 0$ for each fixed x.

Conclusion: $\sin x$ is represented by its Taylor series

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

③ Taylor series for $\cos(x)$: We take $x=0$ as the center.

We can follow the same argument as $\sin(x)$ or use the fact that we can take derivatives term-by-term:

$$\cos x = (\sin x)' = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)' = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \right)'$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

with ROC $= +\infty$ & the series representing $\cos x$ is the Taylor series of $\cos x$, by uniqueness

Note: series has only EVEN powers of x, again consistent with $\cos x$ being an even function.

④ Taylor series for e^{-x^2}:

We can substitute z by $-x^2$ in the series for e^z

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

has ROC $= +\infty$ so it must be the Taylor series for e^{-x^2}.

Application: We can use this to approximate $\int_0^1 e^{-x^2} dx$ by integrating the series term-by-term

$$\int_0^1 e^{-x^2} dx = \sum_{n=0}^{\infty} (-1)^n \int_0^1 \frac{x^{2n}}{n!} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)} \Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} = \frac{1}{3} - \frac{1}{2!\,5} + \frac{1}{3!\,7} - \frac{1}{4!\,9} + \cdots$$

Pick a partial sum as an approximate value for $\int_0^1 e^{-x^2} dx$

Q: What about other centers?

(5) Taylor series for e^{x+1}:

- If the center is -1, we can do the same substitution trick:

$$e^{x+1} = \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$$

- If the center is 0, we write $e^{x+1} = e^x e$

So $e^{x+1} = e\, e^x = e \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} e \frac{x^n}{n!}$

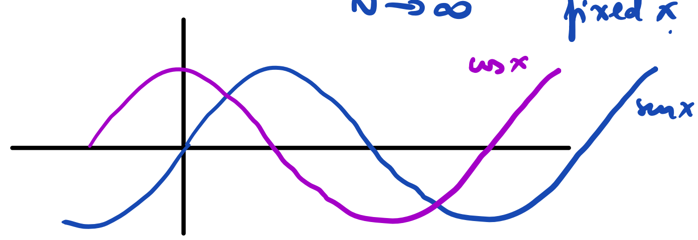
(6) Taylor series for $\sin x$ centered at $x = \frac{\pi}{2}$:

Soln 1: • Compute $\underset{1}{\underset{\parallel}{\sin\frac{\pi}{2}}}$, $\underset{0}{\underset{\parallel}{\cos\frac{\pi}{2}}}$, $\underset{-1}{\underset{\parallel}{-\sin\frac{\pi}{2}}}$, $\underset{0}{\underset{\parallel}{-\cos\frac{\pi}{2}}}$, etc

• Taylor series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}$

• Check ROC $= +\infty$ & $|R_N(\sin x)(x)| \xrightarrow[N\to\infty]{} 0$ for any fixed x.

Soln 2: Write $\sin x = \cos\left(x - \frac{\pi}{2}\right)$



So $\sin x = \cos(\underbrace{x - \tfrac{\pi}{2}}_{=z}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}$

Taylor series for $\cos z$ at $z=0$
substitute $z = x - \frac{\pi}{2}$

§3. Proof of Lagrange Formula:

Pick $c = 0$ as the center & write $Q_N(x) := \frac{R_N(f)(x)}{(x-0)^{N+1}}$ for $x \neq 0$

We want to show that $Q_N(x) = \frac{f^{(N+1)}(b)}{(N+1)!}$ for some b in between 0 & x

To do so, we fix $x \neq 0$ & define a new function $F(t)$ for all t in between 0 & x.

We use $f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \dots + \frac{f^{(N)}(0)}{N!}(x-0)^N + R_N(f)(x)$

$$= f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \dots + \frac{f^{(N)}(0)}{N!}(x-0)^N + Q_N(x)(x-0)^{N+1}$$

We define $F(t)$ by replacing every occurrence of 0 in the (RHS) by t & taking the difference between $f(x)$ & this expression:

$$F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \dots - \frac{f^{(N)}(t)}{N!}(x-t)^N - Q_N(x)(x-t)^{N+1}$$

Note
- $F(0) = 0$
- $F(x) = f(x) - f(x) = 0$
- F is continuous on $[0,x]$ because $f(t), f'(t), \dots, f^{(N)}(t)$ are all continuous & $(x-t), (x-t)^2, \dots, (x-t)^{N+1}$ are as well.
- F is differentiable on $(0,x)$ for the same reasons.

By the Mean Value Theorem applied to F we can find b in $(0,x)$ with

$$F'(b) = 0 = \frac{F(x) - F(0)}{x-0}$$

All that remains is to compute F'. We look at small values of N & try to get a pattern:

$N=1$ $\quad F(t) = f(x) - f(t) - f'(t)(x-t) - Q_1(x)(x-t)^2$

$$F'(t) = -f'(t) - f''(t)(x-t) - f'(t)(-1) - Q_1(x)\, 2\cdot(x-t)(-1)$$

$$= \left(-f''(t) + Q_1(x)\, 2\right)(x-t)$$

So $\quad 0 = F'(b) = \left(-f''(b) + 2Q_1(x)\right)\underbrace{(x-b)}_{\neq 0}$ forces

$$0 = -f''(b) + 2Q_1(x) \rightsquigarrow Q_1(x) = \frac{f''(b)}{2}.$$

N=2 $F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2}(x-t)^2 - Q_2(x)(x-t)^3$

$$F'(t) = \cancel{-f'(t)} \cancel{- f''(t)(x-t)} - \cancel{f'(t)(-1)} - \frac{f'''(t)}{2}(x-t)^2$$
$$\cancel{-\frac{f''(t)}{2} 2(x-t)(-1)} - Q_2(x)\, 3(x-t)^2(-1).$$
$$= \left(-\frac{f'''(t)}{2} + 3Q_2(x)\right)(x-t)^2$$

So $0 = F'(b) = \left(-\frac{f^{(3)}(b)}{2} + 3Q_2(x)\right)\underbrace{(x-b)^2}_{\neq 0}$ forces

$$0 = -\frac{f^{(3)}(b)}{2} + 3Q_2(x) \rightsquigarrow Q_2(x) = \frac{f^{(3)}(b)}{3!}$$

For general N, we'll get similar cancellations that give a simple formula for $F'(t)$, namely:

$$F'(t) = -\frac{f^{(N+1)}(t)}{N!}(x-t)^N + (1+N)Q_N(x)(x-t)^N = \left(-\frac{f^{(N+1)}(t)}{N!} + (N+1)Q_N(x)\right)(x-t)^N$$

So $0 = F'(b) = \left(-\frac{f^{(N+1)}(b)}{N!} + (N+1)Q_N(x)\right)\underbrace{(x-b)^N}_{\neq 0}$, forcing

$$0 = -\frac{f^{(N+1)}(b)}{N!} + (N+1)Q_N(x) \rightsquigarrow Q_N(x) = \frac{f^{(N+1)}(b)}{(N+1)!}$$

Conclusion: $R_N(f)(x) = Q_N(x)\, x^{N+1} = \frac{f^{(N+1)}(b)}{(N+1)!} x^{N+1}$ for $x \neq 0$.

- The formula for an arbitrary center c follows by changing variables. Writing $z = x - c$ the center c for x becomes the center 0 for z.

$f(x) = g(z+c)$ $\qquad R_N(g)(z) = \frac{g^{(N+1)}(\tilde{b})}{(N+1)!} z^{N+1}$ for $\tilde{b}$ in between 0 & z

Now $f^{(N+1)}(x) = g^{(N+1)}(z+c)$ & $R_N(f)(x) = R_N(g)(z)$, so we get

$$R_N(f)(x) = \frac{f^{(N+1)}(b)}{(N+1)!}(x-c)^{N+1}$$ & $b = \tilde{b} + c$ in between c & x.