Lecture LIX: §14.5 Computations with Taylor series
§14.6 Applications to differential equations

Recall: $f(x) = \underbrace{f(c) + \frac{f'(c)}{1!}(x-c) + \cdots + \frac{f^{(N)}(c)}{N!}(x-c)^N}_{N^{\text{th}} \text{ Taylor polynomial for } f \text{ with center } c} + R_N(f)(x)$

Lagrange Remainder formula: $R_N(f)(x) = \frac{f^{(N+1)}(b)}{(N+1)!}(x-c)^{N+1}$ for some b in between c & x.

TODAY'S GOAL: Use $R_N(f)(x)$ to approximate values of $f(x)$ for x near c.

§1. Exponentials:

We aim to estimate e up to m decimal places (m is fixed)

Write $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x so $e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$

Q: How far do we need to go to get $e = \sum_{n=0}^{N} \frac{1}{n!}$ up to m decimal places?

A: Need to estimate $e - \sum_{n=0}^{N} \frac{1}{n!} = R_N(e^x)(1)$ ($= R_N(1)$ for short)

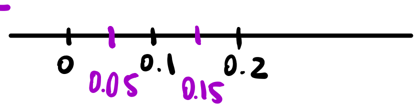
Def: Correct up to m decimal places means $\boxed{|R_N(1)| < 0.5 \cdot 10^{-m}}$

↳ Level of accuracy

Examples $m=1$ gives an error of at most 0.05 ⇝ "1st decimal place is correct"

$m=2$ ———————— 0.005 ⇝ 1st 2 decimal places are correct.

IDEA:



Key $0.1 = 0.0999\cdots = \frac{9}{10^2} + \frac{9}{10^3} + \cdots = \frac{9}{10^2}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right)$

$= \frac{9}{10^2}\left(\frac{1}{1-1/10}\right) = \frac{9}{10^2}\,\frac{10}{9} = \frac{1}{10}$ ✓

In general: $x = k + 0.a_1a_2\cdots a_m a_{m+1}\cdots$

Approx $x = k + 0.a_1\cdots a_m b_{m+1}\cdots$

So $|x - \text{Approx } x| = |0.\underbrace{0\cdots0}_{m \text{ spots}}(a_{m+1}-b_{m+1})\cdots| = \left|\frac{(a_{m+1}-b_{m+1})}{10^{m+1}} + \frac{(a_{m+2}-b_{m+2})}{10^{m+2}} + \cdots\right|$

$= \frac{1}{10^{m+1}}\left|(a_{m+1}-b_{m+1}) + \frac{(a_{m+2}-b_{m+2})}{10} + \cdots\right| \underset{|a_i-b_i|\leq 9}{\leq} \frac{1}{10^{m+1}}\left(9 + \frac{9}{10} + \frac{9}{10^2} + \cdots\right)$

$= \frac{1}{10^{m+1}}\left(9 \cdot \frac{1}{1-1/10}\right) = \frac{1}{10^m}.$

Because of the issue of non-uniqueness of decimal expansions of natural numbers with finite decimal expansion, we need to use 1 more digit

Level of Accuracy: $k+0.a_1\cdots a_m a_{m+1}(a_{m+2}-1) \le x \le k+0.a_1\cdots a_m a_{m+1}(a_{m+2}+1)$

(Rounding error = 0.5) $k+\frac{a_1}{10}\cdots\frac{a_{m+1}}{10^{m+1}}+\frac{a_{m+2}-1}{10^{m+2}} \le x \le k+\frac{a_1}{10}+\cdots+\frac{a_{m+1}}{10^{m+2}}\ \frac{a_{m+2}+1}{10^{m+2}}$

difference $= \frac{2}{10^{m+2}} = \frac{5}{10^{m+1}} = 0.5\frac{1}{10^m}$

- Back to our example:

Want $|R_N(1)| = \left|\frac{e^b}{(N+1)!}\ 1^{N+1}\right| = \frac{e^b}{(N+1)!} < 0.5\ 10^{-m}$.

Since $0<b<1$, then $e^b < e = 2.7182\cdots < 3$

Thus, it's enough to require $\frac{3}{(N+1)!} < 0.5\ 10^{-m}$ To ensure $|R_N(1)|<\frac{0.5}{10^m}$

$$\boxed{10^m < \frac{(N+1)!}{6}}$$

TABLE:

N	$\frac{(N+1)!}{6}$	$\lfloor \log_{10}\frac{(N+1)!}{6}\rfloor$
1	1/3	0
2	1	0
3	4	0
4	20	1
5	120	2
6	840	2
7	6,720	3
8	60,480	4

For $m=1$, need $10<\frac{(N+1)!}{6}$ so $N=4$ works.

$e \sim 1+1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!} = \frac{65}{24} = 2.7083$

For $m=2$ need $100<\frac{(N+1)!}{6}$ so $N=5$ works

$e \sim 1+1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!} = 2.7106$

For $m=3$, need $1000<\frac{(N+1)!}{6}$ so $N=7$ works.

$e \sim 1+1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!} = 2.71825396 83$

For $m=4$, need $10000<\frac{(N+1)!}{6}$. so $N=8$ works

$e \sim 1+1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!} = 2.718278 7699$

In general: $e \sim 1+1+\frac{1}{2!}+\cdots+\frac{1}{N!}$ gives $m=\left\lfloor \log_{10}\frac{(N+1)!}{6}\right\rfloor$ digit accuracy

§2 Sines & Cosines

Same idea: want $|R_N(f)_{(x)}| \leq 0.5\ 10^{-m}$ where m is fixed to get the first m digits of $f(x)$ correctly computed from the N^{th} Taylor polynomial

EXAMPLE 1: Approximate $\cos 93^\circ$ up to 6 decimal places:

Use Taylor polynomial with center $90^\circ = \frac{\pi}{2}$. Set $x = \frac{31}{60}\pi \ (= 93^\circ)$

Last Time: $\sin x = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}$

This series has ROC $= +\infty$ by Ratio Test applied to $|a_n| = \frac{1}{(2n)!}\left|x - \frac{\pi}{2}\right|^{2n}$.

Term-by-Term differentiation gives:

$$\cos x = -\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x - \frac{\pi}{2}\right)^5 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}\left(x - \frac{\pi}{2}\right)^{2n-1}$$

$$|R_N(\cos(x))_{(x)}| = \frac{|(\cos x)^{(N+1)}(b)|}{(N+1)!}\left|x - \frac{\pi}{2}\right|^{N+1}$$

$\cos(x)^{(N+1)} = \pm \sin x$ or $\pm \cos x$ so $|(\cos x)^{(N+1)}_{(b)}| \leq 1$ for any b.

$$|R_N(\cos(x))_{(\frac{31}{60}\pi)}| \leq \frac{1}{(N+1)!}\left|\frac{31}{60}\pi - \frac{\pi}{2}\right|^{N+1} = \left(\frac{\pi}{60}\right)^{N+1}\frac{1}{(N+1)!}$$

To get $|R_N(\cos(x))_{(\frac{31}{60}\pi)}| < 0.5\ 10^{-6}$ is enough to require

$$\left(\frac{\pi}{60}\right)^{N+1}\frac{1}{(N+1)!} < 0.5 \cdot 10^{-6}. \qquad N=3 \text{ does the trick.}$$

Consequence: $\cos 93^\circ = -\frac{\pi}{60} + \frac{1}{3!}\left(\frac{\pi}{60}\right)^3 \simeq -0.\underline{052336}\ldots$

6 digit accuracy

§3. Application to differential equations:

- **INPUT**: A differential equation
- **OUTPUT**: A power series with ROC $= R$ (>0 or $+\infty$, ideally) solving the equation
- **STEPS**: 1. Propose a solution $\sum_{n=0}^{\infty} a_n x^n$ (with ROC $= R \neq 0$)

  2. Differentiate term-by-term to find a recursive relation among

the a_n's using the input differential equation.

3. Write down the series & check if $R \neq 0$. If not, then the output is "no power series can give a solution".

Note: If we are lucky, we may recognize the solution as an elementary function

EXAMPLES:

① $y'_{(x)} = y_{(x)}$ $\rightsquigarrow$ 1. $y(x) = \sum_{n=0}^{\infty} a_n x^n$

2. $y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^3 + \cdots$

$= \sum_{n=0}^{\infty} a_{n+1}(n+1) x^n$

Since $y' = y$ we get $a_0 = a_1$

$a_1 = 2a_2 \rightsquigarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2}$

$a_2 = 3a_3 \rightsquigarrow a_3 = \frac{a_2}{3} = \frac{a_0}{3!}$

$\vdots$

$a_n = (n+1)a_{n+1} \rightsquigarrow a_{n+1} = \frac{a_n}{(n+1)} = \frac{a_0}{(n+1)!}$

So $y(x) = a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{3!} x^3 + \cdots = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is the proposed soln with a_0 an arbitrary number.

3. Check ROC $= +\infty$ by Ratio Test $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \Big/ \frac{|x|^n}{n!} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$

We recognize this as the Taylor series for $a_0 e^x$.

So general solution $y(x) = a_0 e^x$ for any a_0 in $\mathbb{R}$

② $y''(x) + y(x) = 0$ Simple Harmonic Motion

1. $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

2. $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y''(x) = 2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3\, a_4 x^2 + \cdots = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$

From $y''(x) = -y(x)$ we get $a_n = -(n+1)(n+2)a_{n+2}$ for all n.

So $2a_2 = -a_0 \rightsquigarrow a_2 = -\frac{a_0}{2}$

$3\cdot 2 a_3 = -a_1 \rightsquigarrow a_3 = -\frac{a_1}{3!}$

$4\cdot 3\, a_4 = -a_2 \rightsquigarrow a_4 = \frac{-a_2}{4\cdot 3} = \frac{a_0}{4!}$

$5\cdot 4\, a_5 = -a_3 \rightsquigarrow a_5 = \frac{-a_3}{5\cdot 4} = \frac{a_1}{5!}$

$6\cdot 5\, a_6 = -a_4 \rightsquigarrow a_6 = \frac{-a_4}{6\cdot 5} = \frac{-a_0}{6!}$

$7\cdot 6\, a_7 = -a_5 \rightsquigarrow a_7 = -\frac{a_5}{7\cdot 5} = \frac{-a_1}{7!}$

We see the value of a_0 determines $a_2, a_4, a_6, \dots$

——— a_1 ——— $a_3, a_5, a_7, \dots$

Even coefficients $a_{2n} = (-1)^n \frac{a_0}{(2n)!}$ $(n \geq 0)$

Odd coefficients: $a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}$ $(n \geq 0)$

We get $y(x)$ is the sum of 2 series:

$$y(x) = a_0 \underbrace{\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \right)}_{=:\, y_1(x)} + a_1 \underbrace{\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right)}_{=:\, y_2(x)}$$

By Ratio Test, both $y_1(x)$ & $y_2(x)$ have RoC $= +\infty$ so they converge abs. over $\mathbb{R}$ & so will $y(x) = a_0 y_1(x) + a_1 y_2(x)$. This implies $y(x)$ has RoC $= +\infty$ as well (convergence is absolute on $\mathbb{R}$, so no point where it diverges).

But $y_1 = \cos x$ & $y_2 = \sin x$.

Thus, the solution to SHM is a linear combination of $\sin(x)$ & $\cos(x)$.

3 Bessel's Equation $xy'' + y' + xy = 0$

1. $y(x) = \sum_{n=0}^{\infty} a_n x^n$

2. $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} x^m = 2a_2 + 3\cdot 2 a_3 x + \cdots$

So $$\begin{aligned} xy''+y'+xy = &\quad 2a_2x + 3\cdot 2a_3x^2 + 4\cdot 3a_4x^3 + \cdots \\ &+ a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots \\ &\quad a_0x + a_1x^2 + a_2x^3 + \cdots \end{aligned}$$

$$0 = a_1 + (4a_2+a_0)x + (9a_3+a_1)x^2 + (16a_4+a_2)x^3 + \cdots$$

Formally:

$$0 = \sum_{m=0}^{\infty}(m+1)(m+2)a_{m+2}x^{m+1} + \sum_{m=0}^{\infty}(m+1)a_{m+1}x^m + \sum_{m=0}^{\infty}a_mx^{m+1}$$

$$= \sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n + \underbrace{\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n}_{\text{isolate } n=0 \text{ Term}} + \sum_{n=1}^{\infty}a_{n-1}x^n$$

$$= \sum_{n=1}^{\infty}\Big(\underbrace{\big(n(n+1)+n+1\big)a_{n+1} + a_{n-1}}_{(n+1)^2a_{n+1}+a_{n-1}}\Big)x^n + 1a_1x^0$$

$$= a_1 + \sum_{n=1}^{\infty}\left((n+1)^2a_{n+1}+a_{n-1}\right)x^n$$

We get $0 = a_1$

$0 = (n+1)^2a_{n+1} + a_{n-1}$ for all $n \geq 1$. $\rightsquigarrow a_{n+1} = \dfrac{-a_{n-1}}{(n+1)^2}$

$a_1 = 0$

$a_2 = \dfrac{-a_0}{2^2}$

$a_3 = \dfrac{-a_1}{3^2} = 0$

$a_4 = \dfrac{-a_2}{4^2} = \dfrac{a_0}{2^24^2} = \dfrac{a_0}{8^2}$

$a_5 = \dfrac{-a_3}{5^2} = 0$

$a_6 = -\dfrac{a_4}{6^2} = -\dfrac{a_0}{6^2 8^2}$

$\rightsquigarrow$

- all ODD coefficients are 0
- Even coefficients:

$$a_{2m} = \frac{(-1)^m a_0}{2^2 4^2 \cdots (2m)^2} = \frac{(-1)^m a_0}{2^{2m}(m!)^2}$$

Solution: $\displaystyle y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Name: Bessel Function of order 0.

Check ROC $= +\infty$ by ratio test on $b_n = \dfrac{|x|^{2n}}{2^{2n}(n!)^2}$.