Lecture LX: § 14.7 Operations with power series
§ A.16 Division of power series

GOAL: Use algebraic manipulations to compute Taylor series of a function f with ROC $R \neq 0$ without explicitly computing all $f^{(n)}(c)$. In particular we will use the following tools:

1. Substitution
2. Product
3. Long Division

KEY: If f is represented by a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ near $x=c$, then this series MUST be the Taylor series of f with center c (Uniqueness Property)

§1. Substitution of one series in another $f(g(x))$:

EXAMPLE: $f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots$ for $|x| < 1$ = ROC of f

Q1 Series for $\frac{1}{1-x^4}$?

A $g(x) = x^4$ & assume $|x^4| < 1$ (ROC of f)

Then $f(x^4) = \frac{1}{1-x^4} = 1 + (x^4) + (x^4)^2 + \cdots = \sum_{n=0}^{\infty} (x^4)^n = \sum_{n=0}^{\infty} x^{4n}$

has ROC $= 1$ $(= \sqrt[4]{1})$

Q2 Series for $\frac{x^5}{1-x^4}$?

A: $x^5 \sum_{n=0}^{\infty} x^{4n} = \sum_{n=0}^{\infty} x^{4n+5}$ also has ROC $= 1$.

Conclusion: (1) $h(x) = \frac{1}{1-x^4}$ has $h^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is not divisible by } 4 \\ n! & \text{otherwise} \end{cases}$

Taylor series for h with center 0 is $\sum_{n=0}^{\infty} x^{4n} = \sum_{m=0}^{\infty} \frac{h^{(m)}(0)}{m!} x^m$

(2) $P(x) = \frac{x^5}{1-x^4}$ has $P^{(n)}(0) = \begin{cases} 0 & \text{if } n \neq 4k+5 \text{ for some } k \geq 0 \\ n! & \text{otherwise} \end{cases}$

Substitution Rule: Fix $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ &

$g(x) = 0 + b_1 x + b_2 x^2 + \cdots$

Then $f(g(x)) = a_0 + a_1(0 + b_1 x + b_2 x^2 + \cdots) + \underbrace{a_2(0 + b_1 x + b_2 x^2 + \cdots)^2}_{a_2(b_1^2 x^2 + 2b_1 b_2 x^3 + \cdots)} + \cdots$

If f has ROC $= R \neq 0$ & g has ROC $= R' \neq 0$, then $f(g(x))$ has a power series expansion whenever $|x| < R'$ & $|g(x)| < R$. This will happen if and only if $|b_0| < R$.

- If $g(0) \neq 0$, we'll be in trouble (constant term would be $a_0 + a_1 b_0 + a_2 b_2^2 + \cdots$ = a series!)

⚠ We see that to write down the power series for $f(g(x))$ we need to know how to multiply series (we'll need $g(x)^n$ for every $n \geq 0$)

§2. Product of Series:

EXAMPLE 1: $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ROC $= R = +\infty$

$g(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$ ROC $= R = +\infty$

Then, $f(x)\,g(x)$ is a power series with ROC $= \min\{\text{ROC}(f), \text{ROC}(g)\} = +\infty$

How? Distribute and collect coefficients for each power of x.

$$\begin{aligned} f(x)\,g(x) = \; & x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && 1 \cdot g(x) \\ + \; & x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots && x\, g(x) \\ + \; & \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^5}{2 \cdot 5!} - \frac{x^9}{2 \cdot 7!} + \cdots && \frac{x^2}{2}\, g(x) \\ + \; & \frac{x^4}{3!} - \frac{x^6}{3!\,3!} + \frac{x^8}{3!\,5!} - \frac{x^{10}}{3!\,7!} + \cdots && \frac{x^3}{3!}\, g(x) \\ + \; & \cdots \end{aligned}$$

Coefficient for $x = 1$ coefficient for $x^3 = \frac{-1}{3!} + \frac{1}{2} = \frac{1}{3}$ ...

Coefficient for $x^2 = 1$ coefficient for $x^4 = \frac{-1}{3!} + \frac{1}{3!} = 0$

In general, we'll get very complicated formulas.

EXAMPLE 2: $f(x) = \ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$ with ROC $= 1$

$g(x) = \frac{1}{x-1} = -(1 + x + x^2 + x^3 + \cdots)$ with ROC $= 1$

$$
\begin{array}{llll}
\text{So } f(x)g(x) = & x + x^2 + x^3 + x^4 + \cdots & & = x\,g(x) \\
+ & \quad\ \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{2} + \cdots & & = \frac{x^2}{2} g(x) \\
+ & \qquad\quad \frac{x^3}{3} + \frac{x^4}{3} + \cdots & & = \frac{x^3}{3} g(x) \\
& \qquad\quad \vdots & & \quad \vdots \\
\hline
& x + (1+\frac{1}{2})x^2 + (1+\frac{1}{2}+\frac{1}{3})x^3 + (1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})x^4 + \cdots & &
\end{array}
$$

So $f(x)g(x) = \sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) x^n$

Q: When can we use this technique?

A: We are rearranging a series (To collect the various terms with the same x^n) so we need $f(x)$ & $g(x)$ To converge absolutely.

Conclusion: It will work if f & g are power series with the same center, $|x| < \min\{\text{ROC}(f), \text{ROC}(g)\}$

Product Rule: Set $f(x) = \sum_{n=0}^{\infty} a_n x^n$ & $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with $R_1 = \text{ROC}(f) > 0$ & $R_2 = \text{ROC}(g) > 0$. Then we get $f(x)g(x)$ by term-by-term multiplication & adding along columns.

$$
\begin{array}{lll}
a_0 g(x) & = & a_0 b_0 + a_0 b_1 x + a_0 b_2 x^2 + a_0 b_3 x^3 + \cdots \\
+\ a_1 x\, g(x) & & \qquad\qquad\quad a_1 b_0 x + a_1 b_1 x^2 + a_1 b_2 x^3 + \cdots \\
a_2 x^2 g(x) & & \qquad\qquad\qquad\qquad\quad a_2 b_0 x^2 + a_2 b_1 x^3 + \cdots \\
\vdots & & \qquad\qquad\qquad\qquad\qquad\quad \vdots \\
\hline
& & a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots
\end{array}
$$

no terms below the staircase

So $f(x)g(x) = \sum_{n=0}^{\infty} \underbrace{\left(\sum_{k=0}^{n} a_k b_{n-k}\right)}_{\text{coefficient for } f\cdot g} x^n$ $= (*)$ & the series converges absolutely if $|x| < R = \min\{R_1, R_2\}$.

Q Why does this work?

A Take partial sums for f & g & multiply them together.

$s_n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$t_n = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$

$\leadsto s_n t_n = \sum_{p=0}^{2n} \sum_{k=0}^{p} a_k b_{p-k} x^p$

$\hookrightarrow a_k = 0$ if $k > n$
$b_{p-k} = 0$ if $p-k > n$

Rearrange $s_n t_n$ as:

$$\begin{array}{l|c|c|c|c}
x^0 & a_0 b_0 & a_0 b_1 & a_0 b_2 & a_0 b_3 & \dots \\
x^1 & a_1 b_0 & a_1 b_1 & a_1 b_2 & a_1 b_3 & \dots \\
x^2 & a_2 b_0 & a_2 b_1 & a_2 b_2 & a_2 b_3 & \\
x^3 & a_3 b_0 & a_3 b_1 & a_3 b_2 & a_3 b_3 & \ddots \\
x^4 & \vdots & & \vdots & &
\end{array}$$

Rows $= a_i x^i g(x)$
Columns $= b_j x^j f(x)$

① Sum the Ls gives $s_n t_n$

② Sum along antidiagonal is the partial sum of the series $(*)$

By absolute convergence, we can rearrange the series in <u>ANY</u> way and get the same sum.

<u>Way</u> ① $s_n t_n \xrightarrow[n\to\infty]{} f(x) g(x)$

<u>Way</u> ② series in $(*)$.

⚠ We really need absolute convergence for this to work:

<u>**EXAMPLE**</u> $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n+1}} = 1 - \frac{x}{\sqrt{2}} + \frac{x^2}{\sqrt{3}} - \dots$

not abs convergent at $x=1$, only conditionally convergent

Series for $f^2(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} \right) x^n$

$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}} \right) x^n$

This series diverges for $x=1$ but $f^2(1)$ exists!

§3 Division of power series

Division Rule: Given $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ & $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$ with $g(0) = b_0 \neq 0$, we can determine $\frac{f(x)}{g(x)}$ via long division. The result will have a positive radius of convergence if f & g do. (Need to avoid zeroes of g!)

To keep things simple, we can always assume $b_0 = 1$.

EXAMPLE $\tan(x) = \frac{\sin x}{\cos x}$ will have a power series expansion with $ROC = \frac{\pi}{2}$ (even though ROC for $\sin x$ & $\cos x = \infty$)

Long division:

$$x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots = \tan(x)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \quad \Big| \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin x$$

$$- \quad x - \frac{x^3}{2} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots$$

$$\frac{x^3}{3} - \frac{1}{30}x^5 + \cdots$$

$$- \quad \frac{x^3}{3} - \frac{x^5}{6} + \cdots$$

$$\frac{2x^5}{15} + \cdots$$

$\vdots$

Alternative: Write the series for $\frac{1}{\cos x}$ & do $\sin x \cdot \frac{1}{\cos x}$ by Product Rule.

Key $= \cos x = 1 + \text{terms in } x$ so we can write $\frac{1}{\cos x}$ as a power series
(Appendix A16)

Proposition If $g(x) = \sum_{n=0}^{\infty} b_n x^n$ has $b_0 \neq 0$ & $ROC > 0$, then $\frac{1}{g(x)}$ has a power series expansion around 0 with $ROC > 0$.

Why? 1. Propose $\frac{1}{g(x)} = \sum_{n=0}^{\infty} c_n x^n$

2. Write $\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = 1$ & find c_n by recursion

since $1 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_k b_{n-k} \right) x^n = b_0 c_0 + (b_0 c_1 + b_1 c_0)x + \cdots$

so const. Term $\quad 1 = b_0 c_0 \rightsquigarrow c_0 = \frac{1}{b_0}$, $b_0 \neq 0$

x-Term $\quad 0 = b_0 c_1 + b_1 c_0 \rightsquigarrow c_1 = \frac{-b_1 c_0}{b_0} = \frac{-b_1}{b_0^2}$

x^2-Term $\quad 0 = b_0 c_2 + b_1 c_1 + b_2 c_0 \rightsquigarrow c_2 = \frac{\boxed{-b_1 c_1 - b_2 c_0}}{b_0}$ already known

$\vdots$

Continuing in this way, we get a formula for each c_n in terms of $b_0, \dots, b_{n-1}$.

3. Show the series $\sum_{n=0}^{\infty} c_n x^n$ has a positive ROC

How? Write $c_n = -\sum_{k=0}^{n-1} \frac{b_{n-k} c_k}{b_0}$ for all $n \geq 1$.

To simplify, we assume $b_0 = 1$ (otherwise, $\frac{1}{g(x)} = \frac{1}{b_0} \underbrace{\frac{1}{(1 + \frac{b_1}{b_0} x + \cdots)}}_{\text{insert this \& check ROC>0}}$)

Since $\sum_{n=0}^{\infty} b_n x^n$ has ROC $= R > 0$, pick $0 < r < R$ & get that

$\sum_{n=0}^{\infty} |b_n| r^n$ converges. This forces $|b_n| r^n \xrightarrow[n \to \infty]{} 0$

In particular, the sequence $\{|b_n| r^n\}_n$ is bounded & we can find $\boxed{K \geq 1}$ (because $b_0 = 1$) with $|b_n| r^n \leq K$ for all n.

$$|b_n| \leq \frac{K}{r^n}$$

Now use $c_n = -\sum_{k=0}^{\infty} b_{n-k} c_k$ for all $n \geq 1$, $c_0 = \frac{1}{1} = 1$

So $|c_0| = 1 \leq K$

$|c_1| = |b_1 c_0| = |b_1| \leq \frac{K}{r}$

$|c_2| = |b_1 c_1 + b_2 c_0| \leq |b_1 c_1| + |b_2 \overset{1}{\overset{\shortparallel}{c_0}}| \leq \frac{K}{r}\frac{K}{r} + \frac{K}{r^2} K = 2\frac{K^2}{r^2}$

$|c_3| = |b_1 c_2 + b_2 c_1 + b_3 c_0| \leq |b_1 c_2| + |b_2 c_1| + |b_3 \underset{1}{\underset{\shortparallel}{c_0}}| \leq \frac{K}{r} \frac{2K^2}{r^2} + \frac{K}{r^2}\frac{K}{r} + \frac{K^3}{r^3} = \frac{4K^3}{r^3} = 2^2 \frac{K^3}{r^3}$

In general: $|c_\ell| \leq 2^{\ell-1} \frac{K^\ell}{r^\ell}$ for all $\ell \geq 1$

Check $|c_{\ell+1}| = |b_1 c_\ell + b_2 c_{\ell-1} + \cdots + b_{\ell+1} c_0| \leq |b_1 c_\ell| + |b_2 c_{\ell-1}| + \cdots + |b_\ell c_1| + |b_{\ell+1}|$
$\leq \frac{K}{r} 2^{\ell-1} \frac{K^\ell}{r^\ell} + \frac{K}{r^2} 2^{\ell-2} \frac{K^{\ell-1}}{r^{\ell-1}} + \cdots + \frac{K}{r^{\ell-1}} \frac{K}{r} + \frac{K^{\ell+1}}{r^{\ell+1}}$

$$= \frac{K^{\ell+1}}{r^{\ell+1}}\left(\underbrace{2^{\ell-1}+2^{\ell-2}+\cdots\cdots+2+1}+1\right) = \frac{K^{\ell+1}}{r^{\ell+1}}\left(\frac{2^{\ell}-1}{2-1}+1\right)$$

$$= 2^{\ell}\,\frac{K^{\ell+1}}{r^{\ell+1}} = 2^{(\ell+1)-1}\,\frac{K^{\ell+1}}{r^{\ell+1}}$$

This confirms our formula.

- This gives: $\sum_{n=0}^{\infty} |c_n|\,|x|^n = 1 + \sum_{n=1}^{\infty} |c_n|\,|x|^n \leq 1 + \sum_{n=1}^{\infty} 2^{n-1}\frac{K^n}{r^n}|x|^n$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2K|x|}{r}\right)^n$$

↑ geometric series

and this series converges if $\left|\frac{2K|x|}{r}\right| < 1$

$|x| < \frac{2r}{K}$ this is true for all $r<R$

meaning, we need $|x| < \frac{2R}{K}$ for absolute convergence of $\sum_{n=0}^{\infty} c_n x^n$

<u>Conclusion</u>: The radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$ is at least $\frac{R}{2K} > 0$.