Lecture LXI: §14.3 (cont.): Differentiation & Integration of power series
Appendix A15: Uniform convergence of power series

§1 Derivatives & Integrals of power series:

THEOREM: Fix $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with ROC $= R > 0$. Then:

(1) f is continuous on $(-R,R)$.

(2) f is differentiable term-by-term on $(-R,R)$ & $f'(x) = a_1 + 2a_2 x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$

(3) f is integrable term-by-term on $(-R,R)$ & $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} + C$.

In particular, these power series all have radius of convergence R (the same!)

Q Why is this true? Let's start with (2) & (3), and show ROC $\geq R$ for both series.

- First attempt at (2): Assume we know the ROC for f is R by the Root Test,

so $\lim_{n\to\infty} |x| \sqrt[n]{|a_n|} = |x| \lim_{n\to\infty} \sqrt[n]{|a_n|}$ < 1 gives convergence

> 1 gives divergence

This says $R = \dfrac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$

(Aside: This assumption is very close to always being true $\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$)

Then $\lim_{n\to\infty} \underbrace{\sqrt[n]{|n a_n x^{n-1}|}}_{= n^{th} \text{ term of the series in (2)}} = \lim_{n\to\infty} \sqrt[n]{n}\; \sqrt[n]{|a_n|}\; |x|^{\frac{n-1}{n}} = \frac{|x|}{R}$

$\sqrt[n]{n} \xrightarrow{n\to\infty} 1$, $\sqrt[n]{|a_n|} \xrightarrow{n\to\infty} \frac{1}{R}$, $|x|^{\frac{n-1}{n}} \xrightarrow{n\to\infty} |x|$

(*) By L'Hôpital Rule: $\ln \sqrt[n]{n} = \frac{1}{n} \ln n$ $\frac{\infty}{\infty}$ So $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$

Then $\lim_{n\to\infty} \sqrt[n]{n} = e^0 = 1$

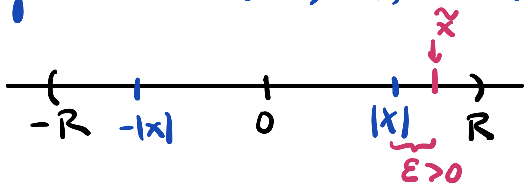
Then: the Root Test says: we converge if $\frac{|x|}{R} < 1$ & diverge if $\frac{|x|}{R} > 1$

Conclude: Radius of convergence for $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is R.

- Second attempt at (2): We show the ROC for (2) is R without exploiting the Root Test for $f(x)$.

We know $\sum_{n=0}^{\infty} |a_n x^n|$ converges if $|x| < R$

Given x in $(-R, R)$, we can always find $\varepsilon > 0$ satisfying $\tilde{x} = |x| + \varepsilon$ lies in $(0, R)$



Then, the series $\sum_{n=0}^{\infty} |a_n| \tilde{x}^n$ converges

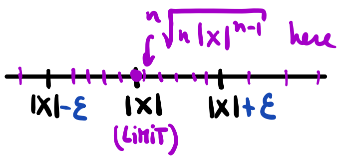
(here we are using $f(\tilde{x})$ converges absolutely because $0 < \tilde{x} < R$)

<u>Claim</u>: $|n x^{n-1}| \leq (|x| + \varepsilon)^n = \tilde{x}^n$ for n large enough

This will say $\sum_{n=n_0}^{\infty} |n a_n x^{n-1}| \leq \underset{\nwarrow \text{ convergent}}{\sum_{n=n_0}^{\infty} |a_n| \tilde{x}^n}$ so by comparison, the tail $\sum_{n=n_0}^{\infty} n a_n x^{n-1}$ will converge absolutely if $|x| < R$, so the full series for (2) will have the same property. This will say <u>ROC for (2) is $\geq R$</u>.

• <u>Proof of the claim:</u> We show $\sqrt[n]{|n x^{n-1}|} \leq |x| + \varepsilon$ for n large enough.

Indeed $\sqrt[n]{|n|\,|x|^{n-1}} = \sqrt[n]{n}\; |x|^{\frac{n-1}{n}} \xrightarrow[n \to \infty]{} |x| < |x| + \varepsilon$



so we know $\sqrt[n]{n|x|^{n-1}} < |x| + \varepsilon$ for n large enough

<u>Only missing point</u>: we still don't know why the series for (2) is f'

• <u>First attempt at (3)</u>: Rewrite $\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} + C$ as $\sum_{n=0}^{\infty} b_n x^n$, so $b_0 = C$, $b_1 = a_0$, $b_2 = \frac{a_1}{2}$, $b_3 = \frac{a_2}{3}$, etc.

In particular $|b_{n+1}| = \frac{|a_n|}{n+1} \leq |a_n|$ for $n \geq 0$. This gives:

$$\sum_{n=0}^{\infty} |b_n| |x|^n = C + \sum_{n=1}^{\infty} |b_n| x^n = C + \sum_{n=0}^{\infty} |b_{n+1}| |x^{n+1}| \leq C + \sum_{n=0}^{\infty} |a_n| |x^{n+1}| =$$
$$= C + |x| \underbrace{\sum_{n=0}^{\infty} |a_n| |x|^n}_{\text{converges for } |x| < R}$$

By comparison we see that $\sum_{n=0}^{\infty} |b_n| |x|^n$ also converges for $|x| < R$.

<u>Conclusion</u>: ROC for series in (3) is $\geq R$.

• Once we confirm that (2) & (3) give the series for f' & $\int f(x)\,dx$, this will force the series to have ROC exactly R (otherwise $\int f'(x)\,dx = f(x) + C$ would

have ROC $\geq$ ROC(f') So, $R = \text{ROC} \geq \text{ROC}(f') \geq R$ gives $\text{ROC}(f') = R$
↑ shown above

Similarly, if $f = (\int f(x)\,dx)'$ has ROC $\geq$ ROC$(\int f(x)\,dx)$

This gives $R = \text{ROC}(f) \geq \text{ROC}(\int f(x)\,dx) \geq R$, forcing $\text{ROC}(\int f(x)\,dx) = R$
↑ shown earlier

Key thing we are using: derivation & integration are "inverse operations" to each other.

- As with (2) we still need to show the series for (3) gives $\int f(x)\,dx$. For both of these statements we'll need the notion of absolute convergence.

§2. Uniform convergence of power series

Recall: $f(x) = \sum_{n=0}^{\infty} a_n x^n = \underbrace{\sum_{n=0}^{N} a_n x^n}_{= S_N(f)_{(x)} = N^{th} \text{ partial sum}} + R_N(f)_{(x)}$

Proposition: If ROC of f is $R > 0$, then $S_N(f)(x) \xrightarrow[N\to\infty]{} f(x)$ if and only if $R_N(f)(x) \xrightarrow[N\to\infty]{} 0$ for every fixed x

In other words, given $\varepsilon > 0$ we can find $N_0 = N_0(x, \varepsilon) > 0$ so that $|R_N(f)_{(x)}| < \varepsilon$ for $N \geq N_0$.

⚠ N_0 in principle depends both on ε & x! We want to remove the dependence on x. This is what yields to the notion of uniform convergence.

Definition: Uniform convergence = N_0 is independent of x as long as x lies in $[-R', R']$ for $R' < R$ (N_0 will depend on ε & R' but will be the same for all x in $[-R', R']$)

More precisely: for each $0 < R' < R$ fixed we have $R_N(x) \xrightarrow[N\to\infty]{} 0$ uniformly on $[-R', R']$, ie $N_0 = N_0(\varepsilon, R')$ is independent on x

Proposition: The series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R', R']$ for every $R' < R = \text{ROC}(f)$

Why? Since $|x| \leq R' < R$ we get $|R_N(x)| = |\sum_{k=N+1}^{\infty} a_k x^k| \leq \sum_{k=N+1}^{\infty} |a_k| \underbrace{|x|^k}_{\leq (R')^k}$

So $|R_N(f)(x)| \leq \sum_{k=N+1}^{\infty} |a_k|(R')^k = \text{Tail of } \sum_{n=0}^{\infty} |a_n|(R')^n$

Since $f(R')$ converges absolutely because $0<R'<R$, we can find $N_0 = N_0(\varepsilon, R')$ where $\left|\sum_{k=N_0+1}^{\infty} |a_k|(R')^k\right| < \varepsilon$.

$=$ | Limit of series $-$ N_0^{th} partial sum |

Combining these 2 facts, we get $|R_N(f)(x)| < \varepsilon$ for $N \geq N_0(\varepsilon, R')$.

- Uniform convergence is a very strong condition. It allows us to finish proving our theorem.

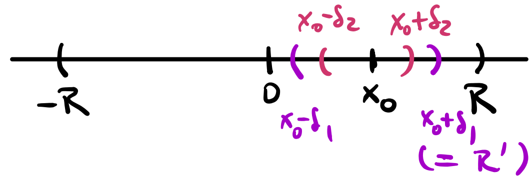
Consequence 1 Power series are continuous (This was item (1) in the Theorem)

Why? Assume we are given x_0 in $(-R,R)$. We want to show $\lim_{x \to x_0} f(x) = f(x_0)$

Pick any $\varepsilon>0$ & write f using N^{th} partial sum $f(x) = \underbrace{S_N(x)}_{\text{polynomial}} + \underbrace{\boxed{R_N(x)}}_{\text{TAIL}}$ $(*)$

Q: What N should we pick?

- Pick $\delta_1>0$ with $-R < x_0-\delta_1 < x_0 < x_0+\delta_1 < R$ ($\delta_1 = \frac{R-|x_0|}{2}$ works)



- Set $R' = \max\{|x_0+\delta_1|, |x_0-\delta_1|\} < R$ by construction

($-R < x_0+\delta_1 < R$ & $-R < x_0-\delta_1 < R$ so $|x_0+\delta_1| < R$ & $|x_0-\delta_1|<R$)

- By uniform convergence, we can find $N_0 = N_0(\varepsilon, R')$ so that $|R_N(y)| < \frac{\varepsilon}{3}$ for every y in $[-R', R']$ & every $N \geq N_0$.

- Pick $N = N_0$ in $(*)$ & write $f(x) = S_{N_0}(x) + R_{N_0}(x)$

If $x \in [-R', R']$, then $|R_{N_0}(x)| < \frac{\varepsilon}{3}$.

Since x_0 is in $[-R', R']$, then $|R_{N_0}(x_0)| < \frac{\varepsilon}{3}$

Now $|f(x) - f(x_0)| = |S_{N_0}(x) + R_{N_0}(x) - (S_{N_0}(x_0) + R_{N_0}(x_0)| =$

$$= \left| (S_{N_0}(x) - S_{N_0}(x_0)) + (R_{N_0}(x) - R_{N_0}(x_0)) \right|$$

regroup

$$\leq |S_{N_0}(x) - S_{N_0}(x_0)| + \underbrace{|R_{N_0}(x)|}_{<\frac{\varepsilon}{3}} + \underbrace{|R_{N_0}(x_0)|}_{<\frac{\varepsilon}{3}}$$

$|a+b| \leq |a| + |b|$

Since S_{N_0} is a polynomial, it is continuous at x_0 so we can find $\delta_2 > 0$ so that $|S_{N_0}(x) - S_{N_0}(x_0)| < \frac{\varepsilon}{3}$

To satisfy all the conditions pick $\delta = \min\{\delta_1, \delta_2\} > 0$. Indeed:

$$|f(x) - f(x_0)| \leq \underbrace{|S_{N_0}(x) - S_{N_0}(x_0)|}_{\substack{<\frac{\varepsilon}{3} \\ (\delta \leq \delta_2)}} + \underbrace{|R_{N_0}(x)|}_{<\frac{\varepsilon}{3}} + \underbrace{|R_{N_0}(x_0)|}_{<\frac{\varepsilon}{3}} < \varepsilon \quad \text{if } |x - x_0| < \delta$$

$(\delta \leq \delta_1)$

We conclude that $\lim_{x \to x_0} f(x) = f(x_0)$.

Consequence 2: Term-by-Term integration

Why? Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has ROC $= R > 0$ & pick $-R < c < d < R$.

Want to show: $\int_c^d \sum_{n=0}^{\infty} a_n x^n \, dx = \sum_{n=0}^{\infty} \int_c^d a_n x^n \, dx = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} \Big|_c^d$

To this end, we again write $\underset{\text{CONT}}{f(x)} = \underset{\text{CONT.}}{S_N(x)} + \underset{\text{CONT.}}{R_N(x)}$ for some suitable N

(by Conseq ①)

All 3 pieces are continuous (use Consequence ①), so we can integrate.

- $S_N(x)$ is a polynomial, so we can integrate term-by-term:

$$\int_c^d \sum_{n=0}^{N} a_n x^n \, dx = \sum_{n=0}^{N} a_n \frac{x^{n+1}}{n+1} \Big|_c^d$$

- $\int_c^d f(x) \, dx = \int_c^d S_N(x) \, dx + \int_c^d R_N(t) \, dx$ (by additivity)

$$= \sum_{n=0}^{N} a_n \frac{x^{n+1}}{n+1} \Big|_c^d + \int_c^d R_N(t) \, dx \overset{?}{=} \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} \Big|_c^d$$

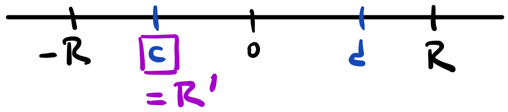
Need to show $\underbrace{\left| \int_c^d R_N(x) \, dx \right|}_{\text{for each } N, \text{ this is a number}} = \left| \text{Tail of } \sum_{n=0}^{\infty} \int_c^d a_n x^n \, dx \right| \xrightarrow[N \to \infty]{} 0$

We show this by uniform convergence!

Given $\varepsilon > 0$, we want to find N_0 with $\left| \int_c^d R_N(x)\,dx \right| < \varepsilon$ if $N \geq N_0$.

Pick $R' = \max\{|c|, |d|\} < R$ & choose N_0 so that $|R_N(x)| < \frac{\varepsilon}{d-c}$ if $N \geq N_0$ for any x in $[-R', R']$

EX:



(we can do this because $R_N(x) \xrightarrow[n\to\infty]{} 0$ uniformly on $[-R', R']$)

Then: $\left| \int_c^d R_N(x)\,dx \right| \leq \int_c^d |R_N(x)|\,dx < \int_c^d \frac{\varepsilon}{d-c}\,dx = \frac{\varepsilon}{d-c}\, x \Big|_c^d = \varepsilon$ if $N \geq N_0$

Conclusion: $\displaystyle \int \sum_{n=0}^{\infty} a_n x^n\,dx = C + \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$

Consequence 3: Term-by-Term differentiation

Why? We want to check that the series $g(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ (with ROC $= R > 0$) represents f'. Since $g(x)$ is continuous on $(-R, R)$ by Consequence ① & we can integrate term-by-term by Consequence ②, we get.

$$\int_0^x g(t)\,dt = \int_0^x \sum_{n=1}^{\infty} n a_n t^{n-1}\,dt = \sum_{n=1}^{\infty} \int_0^x n a_n t^{n-1}\,dt = \sum_{n=1}^{\infty} a_n t^n \Big|_0^x = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0$$

So $f(x) = a_0 + \int_0^x g(t)\,dt$ is differentiable by the Fundamental Theorem of Calculus & $f'(x) = g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ as we wanted to show.