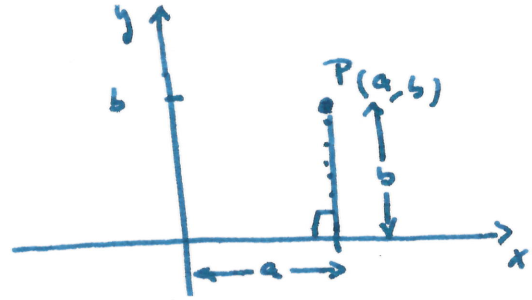
Lecture I (1/11/15) → Section 12.1 (Vectors in the plane)

§1. Two-dimensional coordinate systems:

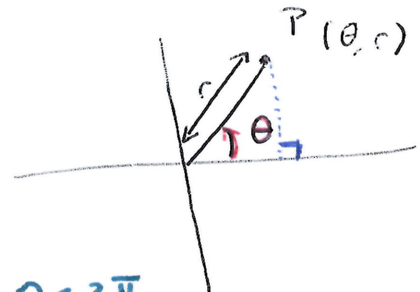
A point in 2-dimensional space ($\mathbb{R}^2$) is completely determined by a pair of numbers:



$P(a, b)$ (a, b in $\mathbb{R}$)

a → x coordinate; b → y-coordinate

We can also characterize P by it's polar coordinates (useful when there are symmetries of objects, e.g. circle)



$P(r, \theta)$ r : radial coordinate $r > 0$

θ angular coordinate

$P(r, \theta + 2\pi) = P(r, \theta)$ ⇝ usually $0 \le \theta < 2\pi$

Conversion rules:

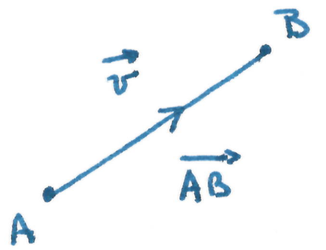
- Cartesian to polar : $r = \sqrt{x^2 + y^2}$; $\begin{cases} \tan\theta = \frac{y}{x} & (x \neq 0) \\ \theta = \frac{\pi}{2} & \text{if } x = 0, y > 0 \\ \theta = -\frac{\pi}{2} & \text{if } x = 0 \ \ y < 0 \end{cases}$

- Polar to cartesian : $x = r\cos\theta$; $y = r\sin\theta$

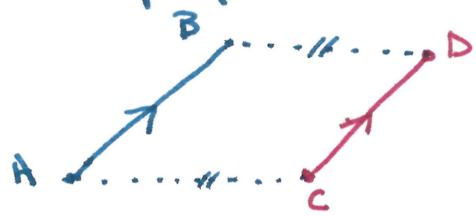
§2. Vectors:

Informally, a vector represents the data of magnitude (or length) and direction. We represent this information by a directed line segment:

- length of the segment = magnitude.
- arrow : indicates the direction.



(1) We do not distinguish between parallel vectors of equal magnitudes



represent the same vector

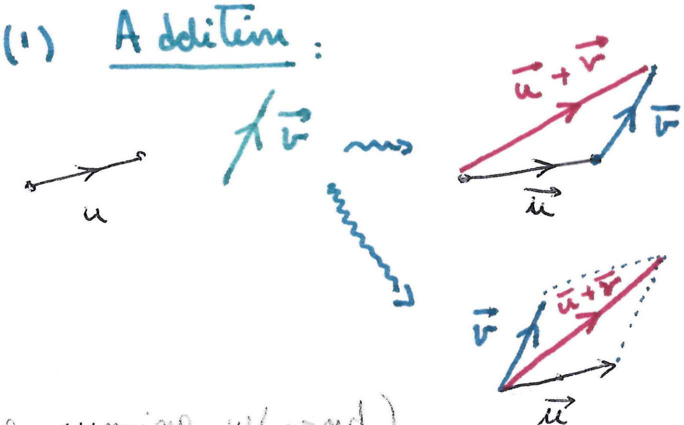
(2) $\vec{0}$ is the unique vector of magnitude 0 (and no direction!)

Notation : $|\vec{v}|$ = magnitude of $\vec{v}$

Note : Quantities having magnitude but no direction are scalars, with the exception of $\vec{0}$. (They are real numbers)

§3. Operations with vectors :

(1) Addition :



[Triangle Law]

[Parallelogram Law]

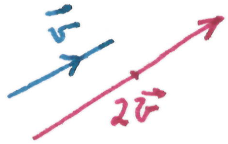
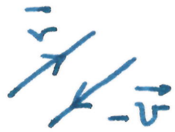
(eg running w/ wind)

(2) Scalar multiplication :

If $\vec{v}$ is a vector and c a scalar (number), then $c\vec{v}$ is a vector determined by :

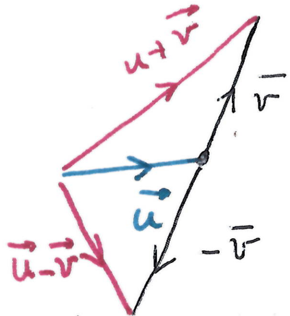
- magnitude of $c\vec{v} = \underbrace{|c|}_{\geq 0} \cdot$ magnitude of $\vec{v}$
- direction of $c\vec{v} = \begin{cases} \text{direction of } \vec{v} & \text{if } c>0 \\ \text{opposite to the direction of } \vec{v} & \text{if } c<0 \end{cases}$

Note : $\vec{v}$ & $c\vec{v}$ are parallel vectors!

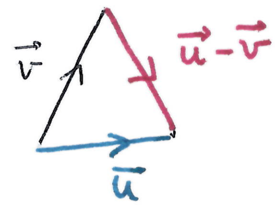
e.g.  , 

(3) Subtraction (difference) : Use (1) + (2) !

If $\vec{u}, \vec{v}$ are two vectors, then $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$



(running against the wind)

Directly : 

(from head of $\vec{v}$ to head of $\vec{u}$)

Properties of Vector Operations

⇝ Axioms for building Linear Algebra (Abstract Vector Spaces)

$\bar{u}, \bar{v}, \bar{w}$ = vectors ; a, c = scalars

(1) $\bar{u} + \bar{v} = \bar{v} + \bar{u}$ (Commutative)

(2) $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$ (Associativity I)

(3) $\bar{v} + \bar{0} = \bar{v}$ ($\vec{0}$ = Additive Identity)

(4) $\bar{v} + (-\bar{v}) = \vec{0}$ (Additive Inverse)

(5) $c(\bar{u} + \bar{v}) = c\bar{u} + c\vec{v}$ (Distributive I)

(6) $(a+c)\vec{u} = a\vec{u} + c\vec{u}$ (" II)

(7) $0\,\vec{v} = \vec{0}$

(8) $c \cdot \vec{0} = \vec{0}$

(9) $1 \cdot \vec{v} = \vec{v}$ (Multiplicative Identity)

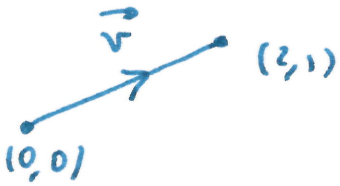
(10) $a(c\vec{v}) = (ac) \cdot \vec{v}$ (Associativity II)

§4. Components of a vector:

To have a consistent (algebraic) representation of vectors, we fix the initial point (tail) of any vector $\vec{v}$ to be the origin.

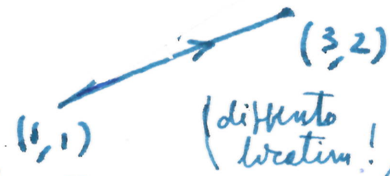
Then: head of $\vec{v}$ = (standard) position vector

eg:



$\rightsquigarrow \quad \vec{v} = \langle 2, 1 \rangle$

same as



(different location!)

Remark: There is a difference between a point $P(a,b)$ and a vector $\vec{v} = \langle a, b \rangle$:

$$\vec{r} = \vec{OP} = \text{position vector of } P \neq P.$$

Property Two position vectors $\vec{u} = \langle u_1, u_2 \rangle$ & $\vec{v} = \langle v_1, v_2 \rangle$ are equal if and only if ($\Longleftrightarrow$) $u_1 = v_1$ & $u_2 = v_2$.

Note: Position of $\vec{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle$ $P = P(p_1, p_2)$, $Q = Q(q_1, q_2)$

Magnitude $P(p_1, p_2)$, $Q(q_1, q_2)$

The magnitude or length of $\vec{PQ}$ is $|\vec{PQ}| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$

The magnitude of a position vector $\vec{r} = \langle a, b \rangle$ is $|\vec{v}| = \sqrt{a^2 + b^2}$.

Operations from components:

(1) Addition: $\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$

(2) Scaler multiplication: $g \langle a_1, a_2 \rangle = \langle g a_1, g a_2 \rangle$

(3) Subtraction $\langle a, b \rangle - \langle c, d \rangle = \langle a - c, b - d \rangle$

§5 Unit vectors:

Defn: A unit vector is any vector of length 1.

Important ones: coordinate unit vectors: $i = \langle 1, 0 \rangle$
$j = \langle 0, 1 \rangle$

- Alternative form for vectors:

$$\vec{v} = \langle a, b \rangle = a\langle 1, 0\rangle + b\langle 0,1\rangle = a\,i + b\,j$$

- Unit vectors in
  - the direction of $\vec{v} \neq \vec{0}$: $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{|\vec{v}|}\cdot\vec{v}$
  - the opposite direction of $\vec{v} \neq \vec{0}$: $\vec{u} = \dfrac{-\vec{v}}{|\vec{v}|} = \dfrac{-1}{|\vec{v}|}\vec{v}$.

Eg: Find all vectors of length 8 parallel to $\langle 3,4\rangle$.

(1) Find 2 unit vectors: $\pm\dfrac{\vec{v}}{|\vec{v}|} = \pm\dfrac{\langle 3,4\rangle}{5} = \pm\langle \frac{3}{5}, \frac{4}{5}\rangle$

(2) Scale by 8 $\Rightarrow$ $\langle \frac{24}{5}, \frac{32}{5}\rangle$ AND $\langle -\frac{24}{5}, -\frac{32}{5}\rangle$
are all the solutions.

§6 Applications: Velocities & Forces. (THURSDAY!)