Lecture II (1/13/16): §12.2 Vectors in 3-dims.

§1. Three-dimensional coordinate systems:

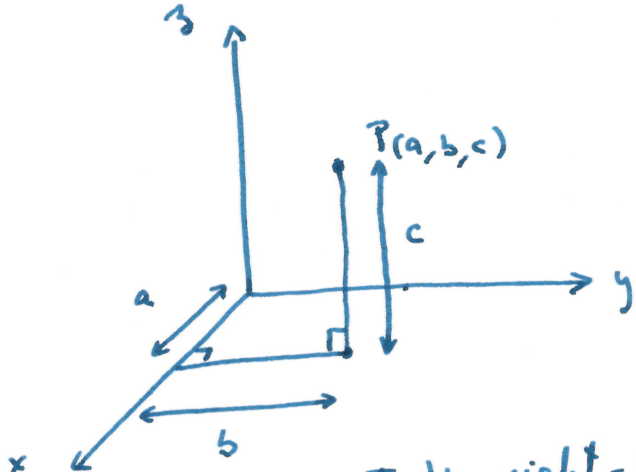
① Cartesian: A point in 3-dimensional space ($\mathbb{R}^3$) is completely determined by a triple of numbers:

$P(a,b,c)$

a = x-coordinate

b = y-coordinate

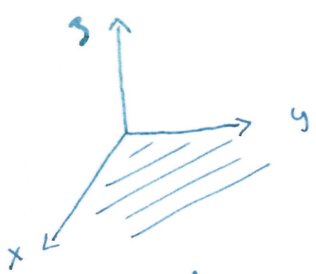
c = z-coordinate



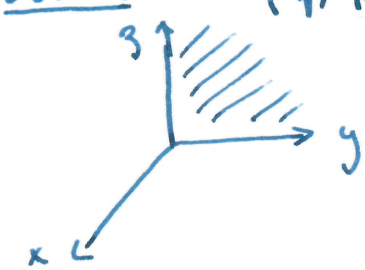
Note: The coordinate axes are directed according to the right-hand rule

- 3 coordinate planes: (i) xy-plane (contains x- and y-axes, equation: $z=0$)
  - (ii) xz-plane (contains x- and z-axes, equ: $y=0$)
  - (iii) yz-plane (contains y- " z-axes, equ. $x=0$)

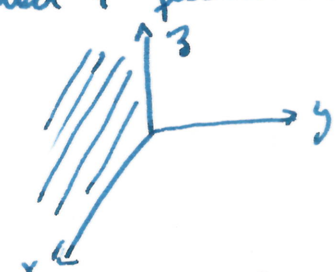
They subdivide $\mathbb{R}^3$ into 8 octants (In $\mathbb{R}^2$ we had 4 quadrants)



xy-plane



yz-plane



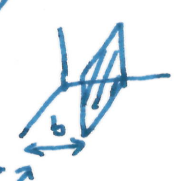
xz-plane

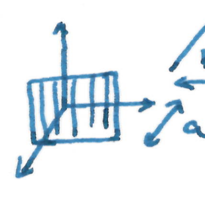
Parallel planes to these have equations: (i) $z=c$

(ii) $y=b$

(iii) $x=a$

↳ translate these planes so that it passed through a point $P(a,b,c)$

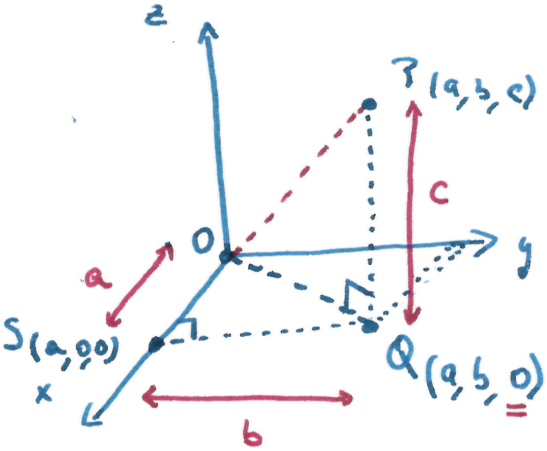




- See 2 more later: Cylindrical ② & Spherical ③. (page 3)

§2 Distance between 2 points in $\mathbb{R}^3$:

Let $P(a,b,c)$ be a point in $\mathbb{R}^3$



Since OPQ is a right angled triangle, we have:

$$(*) \qquad |OP|^2 = |OQ|^2 + \underbrace{|PQ|^2}_{=c^2}$$

Now, OSQ is also a right angled triangle:

$$|OQ|^2 = |OS|^2 + |SQ|^2 = a^2 + b^2$$

Substituting back in (*) gives:

$$|OP|^2 = a^2 + b^2 + c^2$$

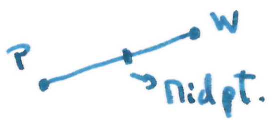
$$\underset{\text{(implies)}}{\Longrightarrow} \quad \boxed{|OP| = \sqrt{a^2+b^2+c^2}}$$

More generally, if $P(a_1, b_1, c_1)$ and $W(a_2, b_2, c_2)$ are two points in $\mathbb{R}^3$, then the distance between P and W is given by the formula

$$|PW| = \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2 + (c_2-c_1)^2}.$$

(Idea: Make W play the role of O)

Applications: ① Midpoint of the line segment joining P and W.



$$\text{Midpoint} = \left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}, \frac{c_1+c_2}{2}\right)$$

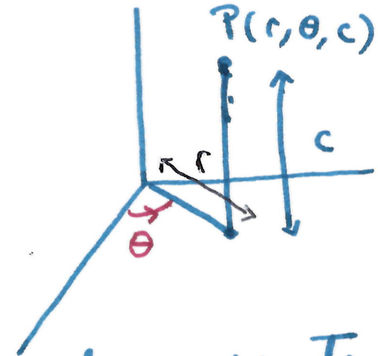
② Sphere: A sphere centered at (a,b,c) with radius r is the set of all points (x,y,z) satisfying $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

③ Ball with center (a,b,c) & radius r: $(x-a)^2+(y-b)^2+(z-c)^2 \leq r^2$

② <u>Cylindrical</u> : Use polar coordinates to describe (x, y)-part.

$P(r, \theta, z)$

- r → radius
- θ → angle in xy-plane from x-axis & counterclockwise
- z → height $= c$



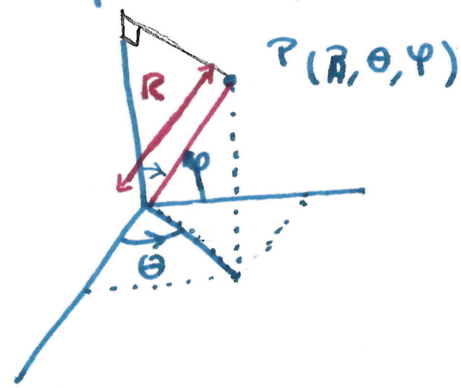
③ <u>Spherical</u>: Use one radial & 2 angular components.

R = radius

θ = angle in xy-plane

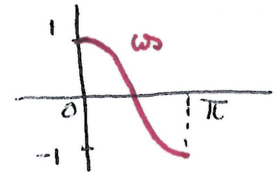
φ = " from z-axis (polar angle)

$0 \leq R$, $\quad 0 \leq \theta < 2\pi$, $\quad 0 \leq \varphi < \pi$



- <u>Conversion rules:</u>
  - Between ① & ② : same as in $\mathbb{R}^2$
  - ③ To ① :
$$\begin{cases} z = R\cos\varphi \\ x = \cos\theta \cdot (R\sin\varphi) \\ y = \sin\theta \cdot (R\sin\varphi) \end{cases}$$
  - ① To ③ :
$$R = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x \neq 0 \\ \frac{\pi}{2} & \text{if } y \geq 0 \\ \frac{3\pi}{2} & \text{if } y < 0 \end{cases}$$

$\varphi = \arccos\left(\frac{z}{R}\right)$ & use the sign of x, y, z to find the value of φ.



§ 3. Vectors in $\mathbb{R}^3$:

Use all ideas & concepts we learned from § 12.1 and add a 3^{rd} coordinate.

Defn: A vector in $\mathbb{R}^3$ is given by a triple of numbers:

$$\vec{v} = \langle a, b, c \rangle$$ → position vector w/ tail at origin $=(0,0,0)$ & head (a,b,c).

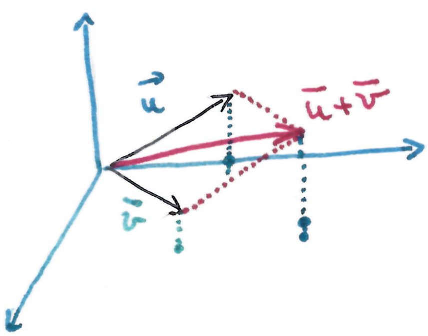
- Magnitude : $|\vec{v}|$ = distance between O & $P(a,b,c)$

$$= \sqrt{a^2+b^2+c^2}$$

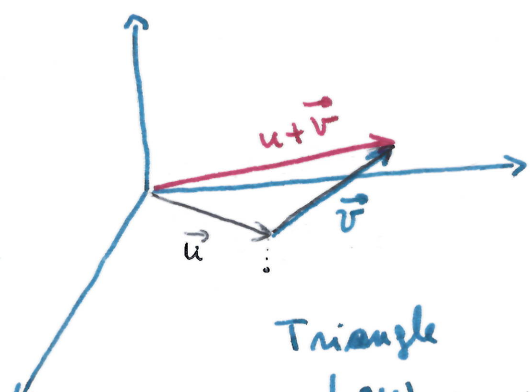
- Vectors with the same magnitude & direction are equal.
- Vector from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ $= \vec{PQ} = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$

  So magnitude of $\vec{PQ}$ = distance from P to Q.
- Operations:

(1) Addition: $\langle a_1, b_1, c_1 \rangle + \langle a_2, b_2, c_2 \rangle = \langle a_1 + a_2, b_1 + b_2, c_1 + c_2 \rangle$



Parallelogram Law



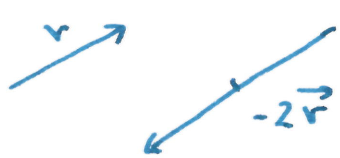
Triangle Law

(2) Scalar Multiplication: $m\langle a, b, c \rangle = \langle ma, mb, mc \rangle$

geometrically: • magnitude of $m\vec{v} = |m| \cdot$ magnitude of $\vec{v}$

• Direction $= \begin{cases} \text{direction of } \vec{v} & \text{if } m > 0 \\ \text{opposite to the direction of } \vec{v} & \text{if } m < 0 \\ \text{no direction } (m\vec{v} = \vec{0}) & \text{if } m = 0 \end{cases}$

Eg.



These operations satisfy 10 properties (see Lecture I)