Lecture III (1/15/16) : §12.3 Dot Product

§1 Definition:

Defn Let $\vec{u} = \langle a, b, c\rangle$, and $\vec{v} = \langle a_2, b_2, c_2\rangle$ be 2 vectors

Then $\vec{u} \cdot \vec{v} := a_1 a_2 + b_1 b_2 + c_1 c_2$ in $\mathbb{R}$ (a number)

is called the dot product of $\vec{u}$ and $\vec{v}$

e.g. $\langle 1, -1, 1\rangle \cdot \langle 0, 3, 5\rangle = 1\cdot 0 + (-1)\cdot 3 + 1\cdot 5 = -3+5 = 2$

Note: A similar definition gives the dot product of vectors in $\mathbb{R}^2$

Properties of the dot product: for $\vec{u}, \vec{v}, \vec{w}$ vectors and m in $\mathbb{R}$ scalar:

(i) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ (Commutative)

(ii) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$ (Distributive)

(iii) $(m\vec{u})\cdot\vec{v} = \vec{u}\cdot(m\vec{v}) = m(\vec{u}\cdot\vec{v})$ (Associative)

(iv) $\vec{v}\cdot\vec{v} = |\vec{v}|^2$.

Geometric form of the dot product:

Let $\vec{u}$ and $\vec{v}$ be two vectors, θ be the angle between them (so $0 \leq \theta \leq \pi$) and $|\vec{u}| = a$, $|\vec{v}| = b$.

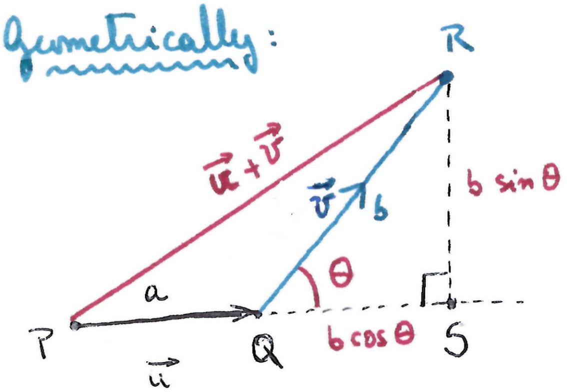
Let us compute $|\vec{u}+\vec{v}|^2$ in two different ways:

- Algebraically:

$$|\vec{u}+\vec{v}|^2 \underset{(iv)}{=} (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \underset{(ii)}{=} \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$\underset{(i)+(iv)}{=} |\vec{u}|^2 + 2\,\vec{u}\cdot\vec{v} + |\vec{v}| = \boxed{a^2+b^2+2\,\vec{u}\cdot\vec{v}} \quad (\mathrm{I})$$

- Geometrically:



$$|\vec{u}+\vec{v}|^2 = |PR|^2$$
$$= |PS|^2 + |RS|^2$$
$$= (a + b\cos\theta)^2 + (b \operatorname{sen}\theta)^2$$
$$= a^2 + 2ab\cos\theta + b^2\cos^2\theta + b^2\operatorname{sen}^2\theta$$
$$(\mathrm{II}) \quad = \boxed{a^2 + 2ab\cos\theta + b^2}$$

sum to 1

Comparing (I) and (II) we get $\vec{u}\cdot\vec{v} = ab\cos\theta$

Conclusion: $\boxed{\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta}$ (True for vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$)

Examples ① If $\vec{u}$ and $\vec{v}$ have lengths 3 and 6 resp., and the angle between them is $60°$, then

$$\vec{u}\cdot\vec{v} = 3\cdot 6\cdot\cos 60° = \frac{18}{2} = 9.$$

② Find the angle between $\langle 2,1,0\rangle$ and $\langle -1,1,5\rangle$

$$\cos\theta = \frac{\langle 2,1,0\rangle\cdot\langle -1,1,5\rangle}{|\langle 2,1,0\rangle|\;|\langle -1,1,5\rangle|} = \frac{-1}{\sqrt{5}\sqrt{27}} = \frac{-1}{3\sqrt{15}} \quad \approx 95°.$$

Note: Two vectors are perpendicular (or orthogonal), that is, the angle between them is $90°$, if and only if their dot product is zero.

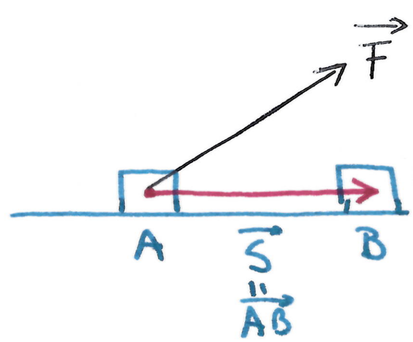
$$\vec{v}\perp\vec{u} \iff \vec{v}\cdot\vec{u} = 0.$$

($\vec{v}$ is perpendicular to $\vec{u}$) (if and only if)

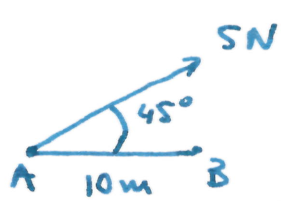
§2. Applications:

① Physics: If a force $\vec{F}$ is acting on an object and the object moves by a displacement vector $\vec{S} = \vec{AB}$, then the work done by the force is given by

$$W = \vec{F}\cdot\vec{S}$$



Example: Find the work done by $\vec{F}$ in moving an object from A to B



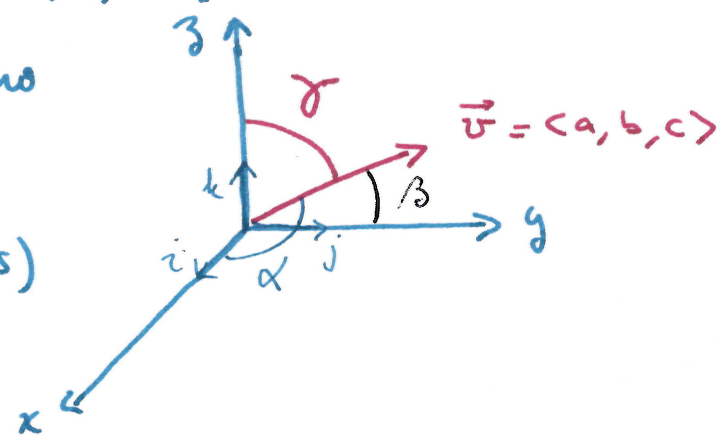
Answer: $5\cdot 10\cdot\cos 45° = 25\sqrt{2}\ \text{Nm} = 25\sqrt{2}\ \text{J}.$

② Direction Angles and direction cosine

Recall: 3 standard (unit) coordinate vectors in $\mathbb{R}^3$

$\vec{i} = \langle 1, 0, 0 \rangle$, $\quad \vec{j} = \langle 0, 1, 0 \rangle$, $\quad \vec{k} = \langle 0, 0, 1 \rangle$

Let $\vec{v} = \langle a, b, c \rangle$ be a non-zero vector



α = angle between $\vec{v}$ and $\vec{i}$ (ie, angle b/w $\vec{v}$ and x-axis)

β = ______ $\vec{v}$ ___ $\vec{j}$ (y-axis)

γ = ______ $\vec{v}$ ___ $\vec{k}$ (z-axis)

Then: $\cos\alpha = \dfrac{\vec{v}\cdot\vec{i}}{|\vec{v}|\,|\vec{i}|}$, $\quad \cos\beta = \dfrac{\vec{v}\cdot\vec{j}}{|\vec{v}|\cdot\vec{j}}$; $\quad \cos\gamma = \dfrac{\vec{v}\cdot\vec{k}}{|\vec{v}|\cdot|\vec{k}|}$

$\Rightarrow \cos\alpha = \dfrac{a}{|\vec{v}|}$, $\quad \cos\beta = \dfrac{b}{|\vec{v}|}$, $\quad \cos\gamma = \dfrac{c}{|\vec{v}|}$

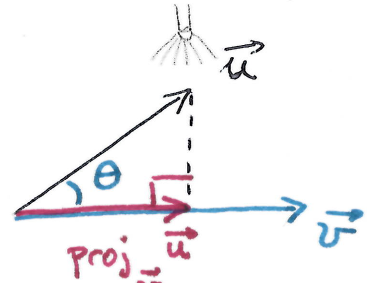
Conclusion: $\langle \cos\alpha, \cos\beta, \cos\gamma \rangle$ is the unique unit vector in the direction of $\vec{v}$ ("direction cosine").

§3. Projections:

Let $\vec{u}$ and $\vec{v}$ be two non-zero vectors. We want to define 2 notions:

- $\text{proj}_{\vec{v}} \vec{u}$:= (orthogonal) projection of $\vec{u}$ onto $\vec{v}$

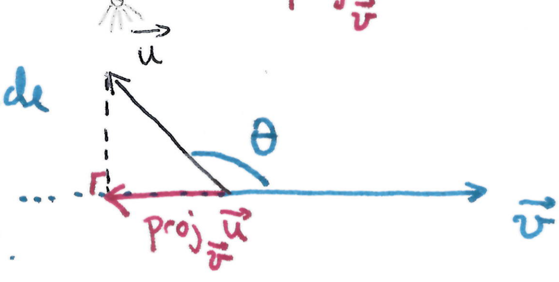
  or vector projection of $\vec{u}$ along $\vec{v}$



$0 \le \theta \le \frac{\pi}{2}$

- $\text{comp}_{\vec{v}} \vec{u}$:= signed magnitude of $\text{proj}_{\vec{v}} \vec{u}$.

  = scalar component of $\vec{u}$ in the direction of $\vec{v}$.



$\frac{\pi}{2} < \theta \le \pi$

Namely, $\text{proj}_{\vec{v}}\vec{u}$ is a vector parallel to $\vec{v}$. The signed magnitude indicates if its direction is equal to the direction of $\vec{v}$ (+ sign) or opposite to the direction of $\vec{v}$ (− sign)

Relation:
$$\text{proj}_{\vec{v}}\vec{u} = (\text{comp}_{\vec{v}}\vec{u}) \cdot \text{unit vector along } \vec{v}$$
$$= \underbrace{(\text{comp}_{\vec{v}}\vec{u})}_{\text{signed length}} \ \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{direction}}.$$

Formula for $\text{comp}_{\vec{v}}\vec{u}$: use right-angled triangle in the 2 figures

$$\text{comp}_{\vec{v}}\vec{u} = |\vec{u}| \cos\theta$$
$$\underset{\text{defn dot prod}}{=} |\vec{u}| \ \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}$$

Note
$$\begin{cases} \text{comp}_{\vec{v}}\vec{u} > 0 & \text{if } 0 \leq \theta < \frac{\pi}{2} \\ \text{comp}_{\vec{v}}\vec{u} = 0 & \text{if } \theta = \frac{\pi}{2} \quad \left(\text{and so } \text{proj}_{\vec{v}}\vec{u} = \vec{0}\right) \\ \text{comp}_{\vec{v}}\vec{u} < 0 & \text{if } \frac{\pi}{2} < \theta \leq \pi \end{cases}$$

Conclusion:
$$\boxed{\text{comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} \quad \text{and} \quad \text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\,\vec{v}}$$

Example: Compute the projections of $\hat{\imath}, \hat{\jmath}, \vec{k}$ along $\langle 1,-1,2\rangle$

- $\text{proj}_{\langle 1,-1,2\rangle}\vec{i} = \dfrac{\langle 1,0,0\rangle \cdot \langle 1,-1,2\rangle}{|\langle 1,-1,2\rangle|^2}\langle 1,-1,2\rangle = \dfrac{1}{6}\langle 1,-1,2\rangle$
- $\text{proj}_{\langle 1,-1,2\rangle}\vec{j} = \dfrac{\langle 0,1,0\rangle \cdot \langle 1,-1,2\rangle}{|\langle 1,-1,2\rangle|^2}\langle 1,-1,2\rangle = \dfrac{-1}{6}\langle 1,-1,2\rangle$
- $\text{proj}_{\langle 1,-1,2\rangle}\vec{k} = \dfrac{2}{6}\langle 1,-1,2\rangle.$

$\text{comp}_{\langle 1,-1,2\rangle}\vec{i} = \dfrac{1}{\sqrt{6}}$; $\text{comp}_{\langle 1,-1,2\rangle}\vec{j} = \dfrac{-1}{\sqrt{6}}$; $\text{comp}_{\langle 1,-1,2\rangle}\vec{k} = \dfrac{2}{\sqrt{6}}.$