Lecture IV (1/20/16) : §12.4 : Cross product

§1. Definition : Let $\vec{u} = \langle a_1, b_1, c_1 \rangle$, $\vec{v} = \langle a_2, b_2, c_2 \rangle$ be two vectors in $\mathbb{R}^3$.

$$\vec{u} \times \vec{v} := \langle b_1 c_2 - b_2 c_1 ,\ \underset{-(a_1c_2 - a_2c_1)}{\overset{\shortparallel}{a_2 c_1 - a_1 c_2}} ,\ a_1 b_2 - a_2 b_1 \rangle$$

is called the cross product of $\vec{u}$ and $\vec{v}$. It's a vector in $\mathbb{R}^3$.

Tip: How to remember the definition? → Use determinants!

Recall : • determinant of a 2x2-matrix

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

• determinant of a 3x3-matrix :

$$\begin{vmatrix} x & y & z \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = x \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - y \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + z \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

With these definitions in mind :

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \begin{array}{l} \text{st.} \\ \vec{u} \\ \vec{v} \end{array} = \vec{i} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Example : $\vec{u} = \langle 1, 2, 3 \rangle$, $\vec{v} = \langle 2, -1, 2 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \vec{i}\,(2 \cdot 2 - (-1) \cdot 3) - \vec{j}\,(1 \cdot 2 - 2 \cdot 3) + \vec{k}\,(1 \cdot (-1) - 2 \cdot 2)$$

$$= \vec{i} \cdot 7 - \vec{j}(-4) + \vec{k}(-5) = \langle 7, 4, -5 \rangle.$$

Properties: $\vec{u}, \vec{v}, \vec{w}$ vectors in $\mathbb{R}^3$, a, b scalars

(i) $\vec{u} \times \vec{v} = - \vec{v} \times \vec{u}$ (Anticommutative)

(ii) $(a\vec{u}) \times (b\vec{v}) = ab\,(\vec{u} \times \vec{v})$ (Associative) (so $\vec{0} \times \vec{v} = \vec{0}$ for all $\vec{v}$)

(iii) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + \vec{u} \times \vec{w}$ (Distributive)

(iv) [Follows from (iii) & (i)] $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

Pf/(i) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$.

Why? **Key Proposition**: The vector $\vec{u}\times\vec{v}$ is perpendicular to <u>both</u> $\vec{u}$ and $\vec{v}$.

<u>Proof</u> We need to verify that $\vec{u}\cdot(\vec{u}\times\vec{v})=0$ (similar for $\vec{v}$)

$$\equiv \langle a_1, b_1, c_1\rangle \cdot \langle b_1c_2-b_2c_1, a_2c_1-a_1c_2, a_1b_2-a_2b_1\rangle$$

$$= a_1(b_1c_2-b_2c_1)+b_1(a_2c_1-a_1c_2)+c_1(a_1b_2-a_2b_1)$$

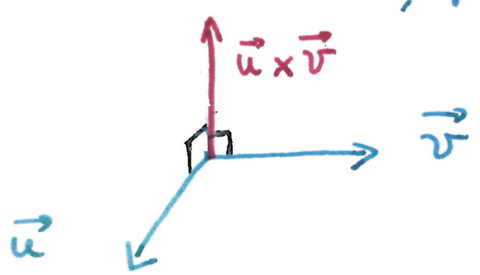
$$= a_1b_1c_2 - a_1b_2c_1 + a_2b_1c_1 - a_1b_1c_2 + a_1b_2c_1 - a_2b_1c_1$$

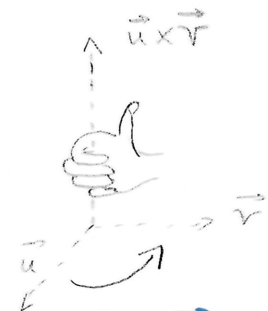
$$=0$$

For $\vec{v}\cdot(\vec{u}\times\vec{v}) \underset{(i)}{=} \vec{v}\cdot(-(\vec{v}\times\vec{u})) = -(\vec{v}\cdot(\vec{v}\times\vec{u}))$

$$= -0 = 0 \checkmark$$

→ <u>Direction</u> of $\vec{u}\times\vec{v}$ is given by the right-hand rule (unless $\vec{u}\times\vec{v}=\vec{0}$, ~~there~~ it has <u>no</u> direction)





<u>Theorem</u> [Geometric definition] Assume $\vec{u}$, $\vec{v}$ are non-zero vectors in $\mathbb{R}^3$ and let θ be the angle between $\vec{u}$ and $\vec{v}$ $(0\le\theta\le\pi)$. Then

$$|\vec{u}\times\vec{v}| = |\vec{u}|\,|\vec{v}|\sin\theta$$

<u>Proof</u> Write $\vec{u}=\langle a_1, b_1, c_1\rangle$, $\vec{v}=\langle a_2, b_2, c_2\rangle$

Verify that $|\vec{u}\times\vec{v}|^2+(\vec{u}\cdot\vec{v})^2 = |\vec{u}|^2|\vec{v}|^2$ (exercise)

Then, since $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$ (Lecture III), we get

$$|\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2(1-\cos^2\theta) = |\vec{u}|^2|\vec{v}|^2\sin^2\theta$$

$$\Longrightarrow |\vec{u}\times\vec{v}| = |\vec{u}|\,|\vec{v}|\,|\sin\theta| \underset{[0\le\theta\le\pi]}{=} |\vec{u}|\,|\vec{v}|\sin\theta$$

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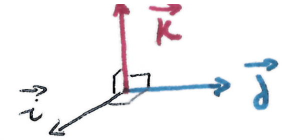
<u>Conclusions</u>: (1) We know the magnitude and the direction of $\vec{u}\times\vec{v}$.

(2) $\vec{u}$ and $\vec{v}$ non-zero vectors are parallel ($\theta=0$ or $180°$) <u>if and only if</u> $\vec{u}\times\vec{v}=\vec{0}$

§2. Consequences:



- **Proposition 1:** (1) $\vec{i} \times \vec{j} = \vec{k}$; (2) $\vec{j} \times \vec{k} = \vec{i}$, (3) $\vec{k} \times \vec{i} = \vec{j}$

  (4) $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$

Proof (1)-(3) All magnitudes are 1 & use right-hand rule.

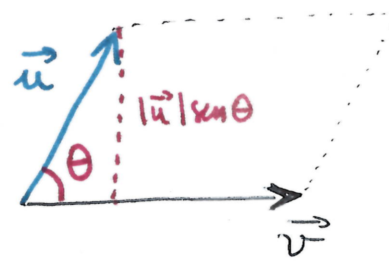
(4) $\vec{i}$ is parallel to $\vec{i}$, so $|\vec{i} \times \vec{i}| = 0$, hence $\vec{i} \times \vec{i} = \vec{0}$ ∎

Note: $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$ in general

Example: $\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -(\vec{k} \times \vec{i}) = -\vec{j}$.

But $(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$.

- **Proposition 2:** The Area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ equals the magnitude of $\vec{u} \times \vec{v}$



Proof Area = height · base = $|\vec{u}|\sin\theta \, |\vec{v}| = |\vec{u} \times \vec{v}|$. ∎

Example 1 Find the area of the parallelogram formed by $\langle 1,0,1\rangle$ and $\langle 2,1,-2\rangle$

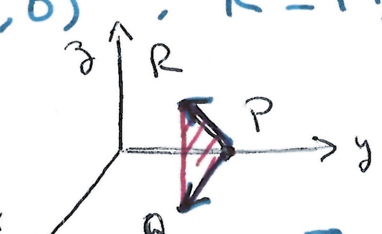
Soln: Area $= |\langle 1,0,1\rangle \times \langle 2,1,-2\rangle|$

$$\langle 1,0,1\rangle \times \langle 2,1,-2\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 2 & 1 & -2 \end{vmatrix} = -\vec{i} - (-4)\vec{j} + \vec{k}$$

$$\Rightarrow \text{Area} = \sqrt{(-1)^2 + (-4)^2 + (1)^2} = \sqrt{18}.$$

Example 2: Find a vector perpendicular to the plane containing

$P = (0,1,0)$, $Q = (1,1,0)$, $R = (1,1,1)$



Soln $\vec{PQ} = \langle 1,0,0\rangle$

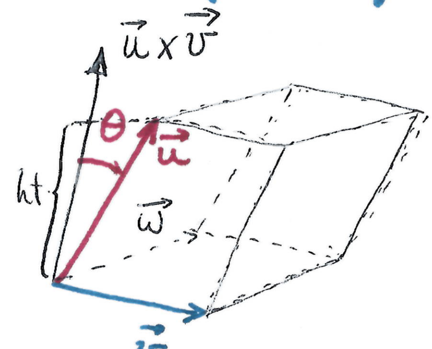
$\vec{PR} = \langle 1,0,1\rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0\vec{i} - \vec{j}(1-0) + \vec{k}\,0 = -\vec{j} = \langle 0,-1,0\rangle$$ ∎

§3. Applications:

I. Volumen of a parallelopipe formed by $\vec{u}$, $\vec{v}$ and $\vec{w}$ is given by

$|\vec{u} \cdot (\vec{v} \times \vec{w})|$ (absolute value)



Why? Area of the base $= |\vec{v} \times \vec{w}|$

Height $= |\vec{u}|\,|\cos\theta|$

where θ = angle between $\vec{u}$ and $\vec{v} \times \vec{w}$

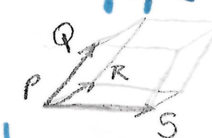
$$\Rightarrow \text{Volume} = |\vec{u}|\,|\vec{v} \times \vec{w}|\,|\cos\theta| = |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

Example: Determine whether the following 4 points lie on a plane (that is, they are coplanar) or not:

$P = (1, 3, 2)$, $Q = (3, -1, 6)$, $R = (5, 2, 0)$ $S = (3, 6, -4)$

Soln: $\vec{PQ} = \langle 2, -4, 4 \rangle$ $\vec{PR} = \langle 4, -1, -2 \rangle$, $\vec{PS} = \langle 2, 3, -6 \rangle$

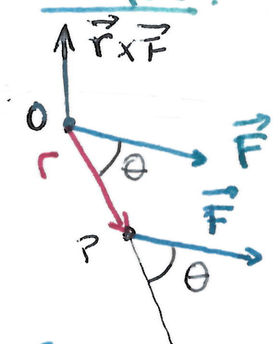
The 4 points are coplanar if and only if the parallelopipe formed by $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$ has volume 0.



$$\text{Vol} = |\vec{PQ} \cdot (\vec{PR} \times \vec{PS})| = \left| \langle 2, -4, 4 \rangle \cdot \begin{vmatrix} i & j & k \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} \right| =$$

$$= |\langle 2, -4, 4 \rangle \cdot \langle 12, 20, 14 \rangle| = 24 - 80 + 56 = 0$$

II. **TORQUE**: Apply a force $\vec{F}$ to a wrench $\vec{r} = \vec{OP}$ at the point P and see the twist effect about the point O.



Twist effect $= \vec{r} \times \vec{F}$.

- magnitude $= |\vec{r}|\,|\vec{F}| \sin\theta$.
- direction = right-hand rule.

So maximum torque if $\theta = 90^\circ$ $(\vec{F} \perp \vec{r})$ & minimum torque: $\theta = 0^\circ$ or $\theta = 180^\circ$

Example: $\vec{r} = \vec{OP} = i - j + 2k$. A force $\vec{F} = \langle 10, 10, 0 \rangle$ is applied at P

Find the torque about O that is produced.

Soln: $\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 10 & 10 & 0 \end{vmatrix} = -20\vec{i} - (-20)\vec{j} + (10+10)\vec{k} = \langle -20, 20, 20 \rangle$

$\Rightarrow$ Torque $= |\vec{r} \times \vec{F}| = 20\sqrt{3}$.