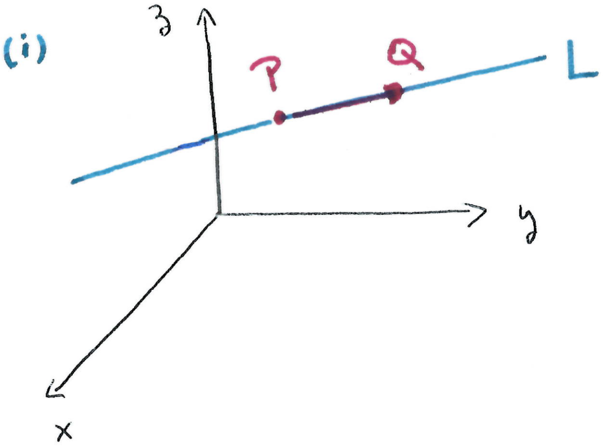
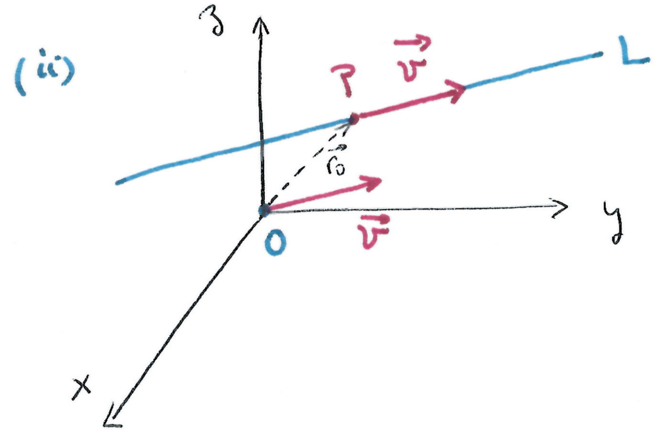
Lecture V (1/22/16). § 12.5 : Lines and curves in space

§ 1. Equation of a line in space:

We can determine a line in 2 ways:



line defined by 2 pts P & Q



line defined by a point P and a direction $\vec{v}$.

How to go between (i) and (ii) : $\vec{v} = \overrightarrow{PQ}$

Equation for L: Let $\vec{r_0} = \overrightarrow{OP}$ Then L is given by the (linear) vector equation

$$\boxed{\vec{r}(t) = \vec{r_0} + t\vec{v} \qquad (t \text{ in } \mathbb{R})}$$

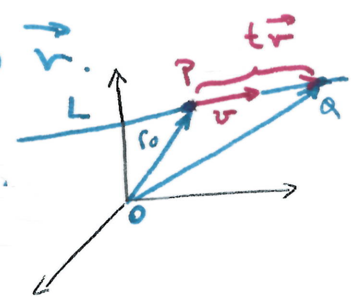
In other words, L consists of all the points Q in $\mathbb{R}^3$ such that

$\overrightarrow{OQ} = \overrightarrow{OP} + t\,\vec{v}$ for some t in $\mathbb{R}$

$\equiv$ Q in $\mathbb{R}^3$ such that $\overrightarrow{OQ} - \overrightarrow{OP} = t\,\vec{v}$

$\equiv$ ______________________ $\overrightarrow{PQ} = t\,\vec{v}$

$\equiv$ ______________________ $\overrightarrow{PQ}$ is parallel to $\vec{v}$.



Parametric equation for L:

Let $P = (x_0, y_0, z_0)$ and $\vec{v} = \langle a, b, c \rangle$

If $Q = (x, y, z)$, then Q lies in L if and only if

$$\underset{(\overrightarrow{OQ})}{\langle x, y, z \rangle} = \underset{(\overrightarrow{OP})}{\langle x_0, y_0, z_0 \rangle} + t\underset{(\vec{v})}{\langle a, b, c \rangle} \quad \text{for some } t$$

$$= \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc \rangle$$

$$\equiv \boxed{x = x_0 + ta \quad , \quad y = y_0 + tb \quad \underline{\text{AND}} \quad z = z_0 + tc} \text{ for } t \in \mathbb{R}$$

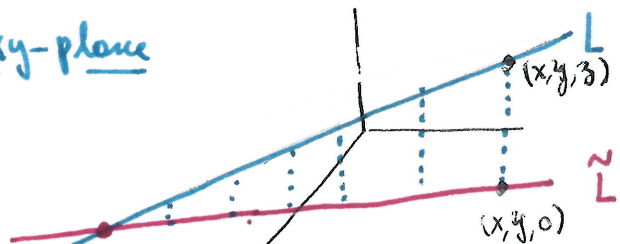
is the parametric form or parametric equation of L.

__Example 1__ Find the equation of the line which is parallel to $\langle 1,2,-1\rangle$ and passes through $(5,0,2)$

__Answer__ $\vec{r}_{(t)} = \langle 5,0,2\rangle + t\langle 1,2,-1\rangle = \langle 5+t, 2t, 2-t\rangle$

Q: How to find projections of lines to the 3 ~~standard~~ coordinate planes?

__Eg: For xy-plane__



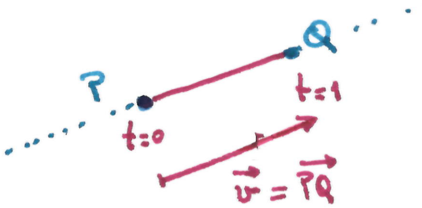
$$\tilde{L} : \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = 0 \quad (\text{xy-plane}) \end{cases}$$

$\Rightarrow$ Eliminate t from these equations gives a relation between x & y

__Eg 1__ (above) $\begin{cases} x = 5 + t & (1) \\ y = 2t & (2) \end{cases} \Longrightarrow t = \frac{y}{2}$ & replace in (1): $\boxed{x = 5 + \frac{y}{2}}$

similarly for yz & xz-planes.

- __Equation of a line segment between two points P & Q.__

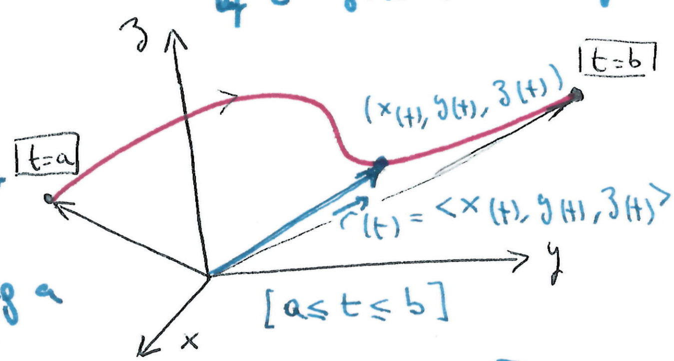


We restrict the equation of the line to specific values of t.

$$\boxed{r_{(t)} = \vec{OP} + t\,\vec{PQ} \quad \text{for } 0 \le t \le 1}$$

Why? For $t=0$ we get $\vec{OP}$ & for $t=1$ we get $\vec{OQ}$ $\rightarrow$ middle values of t give the line segment

__§2 Curves in space:__



We view $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ in two ways:

(1) As a set of parametric equations describing a curve in space

(2) As a vector-valued function when the 3 components of $\vec{r}$ vary with respect to a single independent variable (time t varies between a & b)

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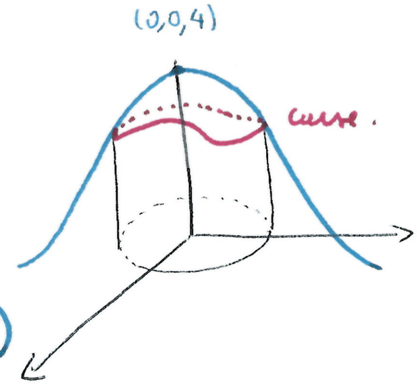
Connection: The point $(x(t), y(t), z(t))$ on the curve is the head of the position vector $\vec{r}(t)$

Examples: ① Lines in space.

② Curves describing intersections of surfaces.

Example: $x^2 + 2y^2 = 4 - z$ and $x^2 + y^2 = 1$.



$$\Rightarrow \quad z = 4 - x^2 - 2y^2 \qquad \left\{ \begin{array}{l} x = \cos\theta \\ y = \sin\theta \end{array} \right.$$

$$z = 4 - \cos^2\theta - 2\sin^2\theta = 3 - \sin^2\theta.$$

$$\vec{r}(\theta) = \langle \cos\theta, \sin\theta, 3 - \sin^2\theta \rangle \qquad (0 \le \theta \le 2\pi)$$

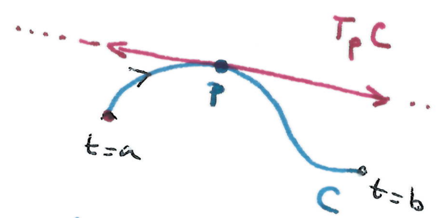
General form: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ for $a \le t \le b$.

Definition: The domain of $\vec{r}(t)$ is the largest set of values of t on which f, g and h are defined.

- Many options for $\vec{r}(t)$:
  - curve can have finite length (Example 2) or extend indefinitely (lines)
  - curve can go from 1 point to another without self-crossing or it can self-cross.
  - it can go in one "direction" or retrace itself ...

§3 Orientation of curves:

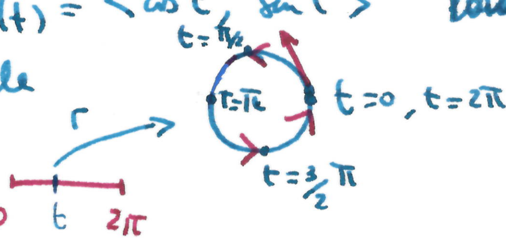
If we view a curve as a set of points, then the curve is smooth if at any point we can draw the tangent lines.



- Viewed as a set, we have 2 tangent directions
- Viewed as a parameterized set we have 1 tangent direction at every point $P = r(t)$ as t moves from a to b increasingly.

This direction defines the positive orientation

Example $r(t) = \langle \cos t, \sin t \rangle$ with $0 \le t \le 2\pi$ parameterizes the unit circle



§4. How to draw curves in space:

TOOL 0: Plot some sample points

TOOL 1: Use projections to 3 coordinate planes, draw the curve in 2-space and try to "lift" the result.

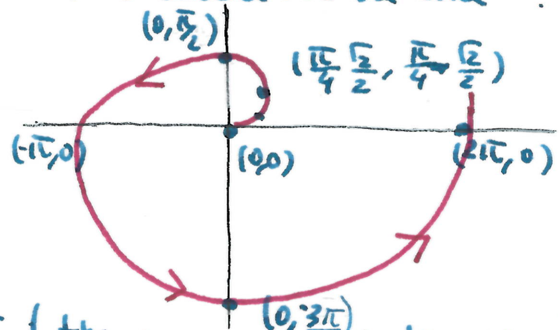
TOOL 2: Find relations (polynomial equations) among the coordinates of $\vec{r}$ to realize the curve as an intersection of 2 surfaces in 3-space.

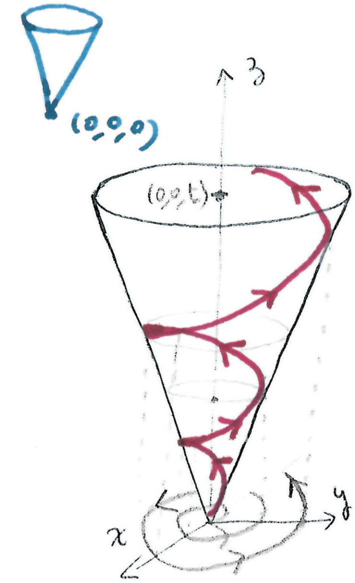
TOOL 3: Detect possible oscillations (eg helix curve [page 835])

- EXAMPLE Draw the curve $r(t) = \langle t\cos t, t\sin t, t \rangle$ for $t \geq 0$

.1 Equation: $x^2 + y^2 = z^2$ This describes an icecream cone.

.Project to the xy-plane:





⇒ Lift each point to height t (the curve lies in the cone)

- More examples in the book!

§5. Limits and Continuity for Vector-Valued Functions:

Upshot: All the notions of calculus for vector-valued functions should be done coordinatewise. Write $r(t) = \langle x(t), y(t), z(t) \rangle$, $L = \langle l_1, l_2, l_3 \rangle$

Why? Definition $\lim\limits_{t\to a} \vec{r}(t) = \vec{L}$ if and only if $\lim\limits_{t\to a} |\vec{r}(t) - \vec{L}| = 0$

- $|\vec{r}(t) - \vec{L}| = \sqrt{(x(t)-l_1)^2 + (y(t)-l_2)^2 + (z(t)-l_3)^2} \xrightarrow[t\to 0]{} 0$

if and only if $x(t) \xrightarrow[t\to a]{} l_1$, $y(t) \xrightarrow[t\to a]{} l_2$ AND $z(t) \xrightarrow[t\to a]{} l_3$

Continuity: $\vec{r}(t)$ is continuous at $t=a$ if and only if $x(t), y(t)$ AND $z(y)$ are continuous at $t=a$.

EXAMPLES (1) Let $r(t) = \langle t, -t, t \rangle$. Then $\vec{r}$ is continuous everywhere

(2) $r(t) = \begin{cases} \langle 1, 0, 0 \rangle & t=0 \\ \langle \frac{\sin t}{t}, t, t \rangle & \text{otherwise.} \end{cases}$ Q: Is $\vec{r}(t)$ continuous at $t=0$?

A: $\lim\limits_{t\to 0} r(t) = \langle \lim\limits_{t\to 0} \frac{\sin t}{t}, \lim\limits_{t\to 0} t, \lim\limits_{t\to 0} t \rangle = \langle 1, 0, 0 \rangle$ so it's continuous at $t=0$.