Lecture VI (1/25/16): §12.6 Calculus of vector-valued functions

Recall: Vector-valued functions $r(t) = \langle f(t), g(t), h(t) \rangle$ for $a \leq t \leq b$

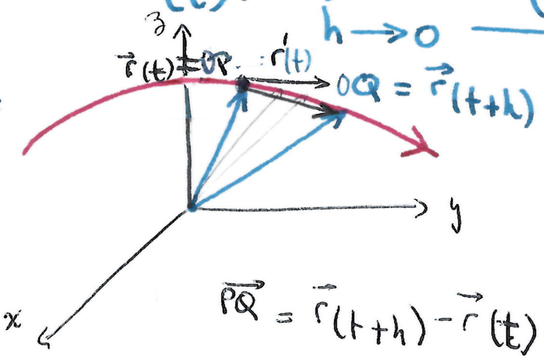
KEY: All notions of Calculus I (limit, continuity, derivatives, integrals) extend to parametric curves by componentwise operations.

§1. The derivative and Tangent vector

• Assume f, g, h are differentiable on $a < t < b$ (so $f'(t), g'(t)$ & $h'(t)$ are well defined for $a < t < b$). Then $\vec{r}(t)$ is differentiable and

Definition: $$\vec{r}\,'(t) := \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle$$

Geometry:



Note: $\vec{r}\,'(t)$ is the Tangent vector to the curve at P = head of $\vec{r}(t)$ (if $\vec{r}\,'(t) \neq \vec{0}$)

Properties of $\vec{r}\,'(t)$:

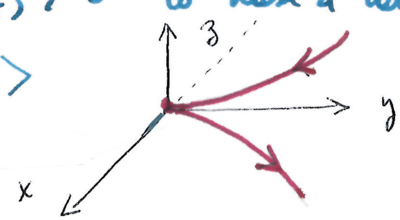
(1). The vector $\vec{r}\,'(t)$ points in the tangent direction of the curve at P

(2) The vector $\vec{r}\,'(t)$ gives the rate of change of $\vec{r}(t)$ at the point P.

Note: If $\vec{r}(t)$ is the position vector of a moving particle, then $\vec{r}\,'(t)$ is the VELOCITY vector of the particle, it always points in the direction of motion, and $|\vec{r}\,'(t)|$ is the SPEED of the particle.

Q: Why do we need $\vec{r}\,'(t) \neq \vec{0}$ to have a tangent vector at $P = \vec{r}(t)$?

Example: $\vec{r}(t) = \langle t^3, t^2, 0 \rangle$



$\vec{r}(t)$ lies on xy-plane

$\vec{r}$ is differentiable and

$\vec{r}\,'(t) = \langle 3t^2, 2t, 0 \rangle$, so $\vec{r}\,'(0) = \vec{0}$

At $t=0$ there's a change in trajectory that creats a cusp or sharp corner at $(0,0,0)$. So there's no tangent direction at $t=0$.

Definition The unit tangent vector for a particular value of t is

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \qquad (\text{provided } \vec{r}'(t) \neq \vec{0}).$$

This vector carries information about the tangent direction.

Example: Find the unit tangent vectors of $\vec{r}(t) = \langle t^4, 4t, 4\ln t\rangle \quad t>0$

Soln: $\vec{r}'(t) = \langle 4t^3, 4, \frac{4}{t}\rangle$

$$|\vec{r}'(t)| = \sqrt{(4t^3)^2 + 4^2 + \left(\frac{4}{t}\right)^2} = 16\sqrt{t^6+1+\frac{1}{t^2}} = \frac{16}{t}\sqrt{t^8+t^2+1}$$

$$\Rightarrow T(t) = \frac{\langle t^3, 4, \frac{4\ln t}{t}\rangle}{16\sqrt{1+t^2+t^8}}$$

§2. Derivative rules: $\vec{u}$, $\vec{v}$ differentiable vector functions in $\mathbb{R}^3$, and $\vec{c} = \langle c_1, c_2, c_3\rangle$ a vector.

Then:

(1) $\frac{d}{dt}(\vec{c}) = 0$ (Constant Rule)

(2) $\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \vec{u}'(t) + \vec{v}'(t)$ (Sum Rule)

(3) $\frac{d}{dt}(f(t)\,\vec{u}(t)) = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$ (Product Rule)

(4) $\frac{d}{dt}(\vec{u}(f(t))) = \underbrace{\vec{u}'(f(t))}_{\text{vector}}\ \underbrace{f'(t)}_{\text{scalar}}$ (CHAIN RULE)

(5) $\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$ (Dot Product Rule)

(6) $\frac{d}{dt}(\vec{u}(t)\times\vec{v}'(t)) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$ (Cross Prod. Rule)

Note: Rules (1)–(4) are the same for functions of 1 variable (Calc I).

Rules (5)–(6) come from the definition of $\cdot$ & $\times$.

Proof 5: $\frac{d}{dt}(\vec{u}\cdot\vec{v}) = \frac{d}{dt}(u_1v_1 + u_2v_2 + u_3v_3) = u_1'v_1 + u_1v_1' + u_2'v_2 + u_2v_2'$
$+ u_3'v_3 + u_3v_3') = \underbrace{(u_1'v_1 + u_2'v_2 + u_3'v_3)}_{=\vec{u}'(t)\cdot\vec{v}(t)} + \underbrace{(u_1v_1' + u_2v_2' + u_3v_3')}_{=\vec{u}(t)\cdot\vec{v}'(t)}$

Conclusion $= \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

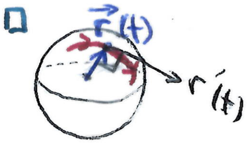
Proposition: $\frac{d}{dt}|\vec{r}(t)| = \frac{\vec{r}\,'(t)\cdot\vec{r}(t)}{|\vec{r}(t)|}$ if $\vec{r}(t) \neq \vec{0}$

Proof: $|\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$ function of 1 variable.

Differentiate on each side & use (5): $2|\vec{r}(t)|\frac{d}{dt}(|\vec{r}(t)|) = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t)$
$= 2\,\vec{r}\,'(t)\cdot\vec{r}(t)$

By assumption $|\vec{r}(t)| \neq 0$, so $\frac{d}{dt}|\vec{r}(t)| = \frac{\vec{r}\,'(t)\cdot\vec{r}(t)}{|\vec{r}(t)|}$. □

Conclusion: If $|\vec{r}(t)|$ is constant, then $\vec{r}(t) \perp \vec{r}\,'(t)$ for every t.



Recall:

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\log x$	$\frac{1}{x}$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x = \frac{1}{\cos^2 x}$

for every $n \neq 0$ (positive or negative number)

§3 Higher derivatives:

We compute higher derivatives componentwise. (iterate derivatives)

Example: $\vec{r}(t) = \langle t^3, \ln t, 4e^{-5t}\rangle$ for $t>0$. Compute $\vec{r}\,'''(t)$.

First compute $\vec{r}\,'(t) = \langle 3t^2, \frac{1}{t}, 4(-5)e^{-5t}\rangle = \langle 3t^2, \frac{1}{t}, -20e^{-5t}\rangle$

Then, $\vec{r}\,''(t) = (\vec{r}\,'(t))' = \langle 6t, \frac{-1}{t^2}, 100e^{-5t}\rangle$

Then, $\vec{r}\,'''(t) = (\vec{r}\,''(t))' = \langle 6, \frac{2}{t^3}, -500e^{-5t}\rangle$

§4. Integrals of vector-valued functions:

Recall: Given a function $f(x)$, the indefinite integral or antiderivative of $f(x)$ is a function $F(x)$ such that $F'(x) := \frac{d}{dx}F(x) = f(x)$.

Note: $F(x)$ is only well-defined up to adding a constant: $\left(\frac{d}{dx}(F(x)+C) = \frac{d}{dx}F(x)\right)$

We write $F(x) = \int f(x)\,dx$.

Recall

	$f(x)$	$\int f(x)\,dx$
$(n \neq 1)$	x^n	$\frac{x^{n+1}}{n+1} + C$
	$\frac{1}{x}$	$\ln x + C$
	e^x	$e^x + C$
	$\sin x$	$-\cos x + C$
	$\cos x$	$\sin x + C$
	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x) + C = \arcsin x + C$