Lecture VII (1/27/16): end of §12.6 & 12.7: Motion in space

§ End of Lecture VI: we discussed indefinite integrals for functions with values in $\mathbb{R}$.

Recall: given $f:[a,b] \longrightarrow \mathbb{R}$, the indefinite integral or antiderivative of f is a function $F:[a,b] \longrightarrow \mathbb{R}$ such that $\frac{dF}{dx} = f(x)$.

Note: F is well-defined & unique up to addition of constants because

$\frac{d}{dx}(F(x)+C) = \frac{d}{dx}F$ for any constant C in $\mathbb{R}$.

We write $F(x) = \int f(s)\,ds + C$

Fundamental Theorem of Calculus: $\underbrace{\int_a^b f(x)\,dx}_{\text{definite integral}} = F(b) - F(a)$ where F is any indefinite integral of f

[Reason: constants cancel out!]

Example: $\int_0^1 e^x\,dx = e^1 - e^0 = e-1$.

- For vector-valued functions: we define antiderivatives & definite integrals componentwise

Definitions: (1) If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ for $a \le t \le b$, the antiderivative of $\vec{r}(t)$ is $\vec{R}(t) = \langle F(t), G(t), H(t)\rangle$ satisfying $\vec{R}'(t) = \vec{r}(t)$.

[F, G, H are antiderivatives of f, g & h, respectively].

Note: $\vec{R}$ is well-defined & unique up to adding any constant VECTOR $\vec{C}$.

We write the indefinite integral of $\vec{r}(t)$ as $\int \vec{r}(s)\,ds = \vec{R}(t) + \vec{C}$.

(2) The definite integral of $\vec{r}(t)$ on $[a,b]$ is $\int_a^b \vec{r}(s)\,ds = \vec{R}(b) - \vec{R}(a)$

Why?

$$\int_a^b \vec{r}(s)\,ds \underset{\text{defn.}}{=} \left\langle \int_a^b f(s)\,ds, \int_a^b g(s)\,ds, \int_a^b h(s)\,ds \right\rangle$$

$$\underset{\text{Fund Thm Calc}}{=} \langle F(b)-F(a), G(b)-G(a), H(b)-H(a)\rangle \underset{\text{rearrange terms}}{=} \vec{R}(b) - \vec{R}(a)$$

Example $\vec{r}(t) = \langle 1-t^2, t, t^3\rangle$ Compute $\int_1^3 \vec{r}(t)\,dt$.

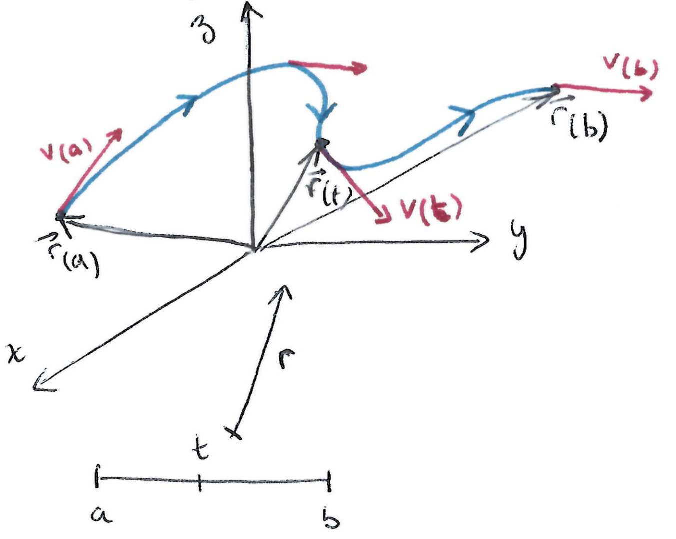
Soln: $\int_1^3 \vec{r}(t)\,dt = \left\langle t - \frac{t^3}{3}, \frac{t^2}{2}, \frac{t^4}{4} \right\rangle \Big|_1^3 = \left\langle 3-9, \frac{9}{2}, \frac{81}{4}\right\rangle - \left\langle \frac{2}{3}, \frac{1}{2}, \frac{1}{4}\right\rangle$

$= \left\langle -\frac{20}{3}, 4, 20\right\rangle$.

§12.7: Motion in Space

§1. Position, velocity, speed and acceleration

In Lecture V we discussed curves in space & their presentation as vector-valued functions of 1 parameter t (time) or by parametric equation. In particular, we realize that the set of points described by the position vectors is different than the function itself (for example, the latter has orientation, but the former doesn't).



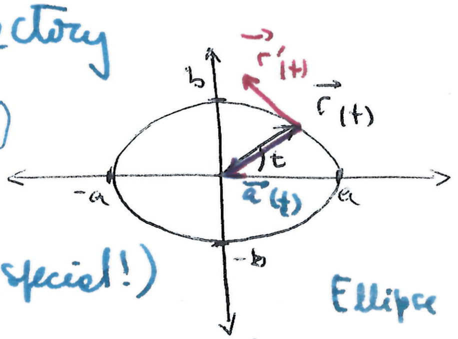
Let $\vec{r}_{(t)}$ be the position vector of a particle at time t (in $\mathbb{R}^2$ or $\mathbb{R}^3$)

- $r(t)$ describes the path or trajectory of the particle

Definitions

- Velocity vector $\vec{v}_{(t)} := \vec{r}\,'(t)$ (it's the instantaneous velocity of the particle at time t)
- Speed = magnitude of the velocity $= |\vec{r}\,'_{(t)}|$ (in $\mathbb{R}_{\geq 0}$)
- Acceleration $= \vec{a}_{(t)} := \vec{v}\,'_{(t)} = \vec{r}\,''_{(t)}$ (rate of change of the velocity vector)

Note: Velocity vectors are tangent vectors to the trajectory



Example 1: $\vec{r}_{(t)} = \langle a\cos t, b\sin t\rangle$ $0 \leq t \leq 2\pi$ $(a, b > 0)$

$\vec{v}(t) = \vec{r}\,'_{(t)} = \langle -a\sin t, b\cos t\rangle$

$\vec{a}(t) = \vec{v}\,'(t) = \langle -a\cos t, -b\sin t\rangle = -\vec{r}(t)$ (special!)

Speed $= |\vec{v}(t)| = \sqrt{(-a\sin t)^2 + (b\cos t)^2}$ $(= a$ if $a = b)$

(Example 2 on next page)

Example 3 Assume $\vec{a}_{(t)} = \langle t, 1-t, t^2\rangle$, and $\vec{v}_{(0)} = \langle 1, 0, 1\rangle$ & $\vec{r}_{(0)} = \langle 0, 0, 0\rangle$

Find the position vector of the particle at time t.

Soln: $\vec{v}\,'_{(t)} = \langle t, 1-t, t^2\rangle$ $\Longrightarrow$ Integrate to get $\vec{v}_{(t)}$.

$$v(t) = \int_0^t \langle s, 1-s, s^2\rangle\, ds + \vec{C} = \langle \frac{t^2}{2}, t - \frac{t^2}{2}, \frac{t^3}{3}\rangle + \vec{C}$$

Use $\vec{v}_{(0)} = \langle 1, 0, 1\rangle$ & $\vec{v}_{(0)} = \vec{0} + \vec{C} = \vec{C}$ to get $\vec{C} = \langle 1, 0, 1\rangle$

Then $\vec{r}\,'_{(t)} = \vec{v}(t) = \langle 1 + \frac{t^2}{2}, t - \frac{t^2}{2}, 1 + \frac{t^3}{3}\rangle$

Integrate again to get $\vec{r}(t) = \int \vec{v}(s)\,ds + \vec{C_2} = \int \langle 1+\frac{s^2}{2}, s-\frac{s^2}{2}, 1+\frac{s^3}{3}\rangle ds + \vec{C_2}$

$= \langle t+\frac{t^3}{6}, \frac{t^2}{2}-\frac{t^3}{6}, t+\frac{t^4}{12}\rangle + \vec{C_2}$

Use $\langle 0,0,0\rangle = \vec{r}(0) = \vec{0} + \vec{C_2} = \vec{C_2}$ to conclude

$$\boxed{\vec{r}(t) = \langle t+\frac{t^3}{6}, \frac{t^2}{2}-\frac{t^3}{6}, t+\frac{t^4}{12}\rangle}$$

Example 2: $\vec{R}(t) = \langle a\cos 2t, b\,\text{sen}\,2t\rangle \qquad 0\leq t\leq \pi$

$\vec{V}(t) = \langle -2a\,\text{sen}\,2t, 2b\cos 2t\rangle$

$\vec{A}(t) = \langle -4a\cos 2t, -4b\,\text{sen}\,2t\rangle = -4\vec{R}(t)$

$\vec{R}(t)$ & $\vec{r}(t)$ describe the same set of points, but $\vec{R}$ goes twice as fast as $\vec{r}$.
$(\vec{R}(t) = \vec{r}(2t))$

§2. Straight-line and circular motion:

① Straight line (see Lecture V) → UNIFORM (CONSTANT VELOCITY)

$$\boxed{\vec{r}(t)} = \underset{\text{(initial pt P)}}{\langle x_0, y_0, z_0\rangle} + t\underset{\text{(direction vector)}}{\langle a,b,c\rangle} = \boxed{\begin{array}{c}\langle x_0+at, y_0+bt, z_0+ct\rangle \\ \text{for } t\geq 0\end{array}}$$

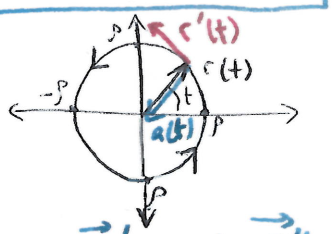
Note: direction $(=\vec{r}\,'(t))$ is constant, so $\vec{a}(t) = \langle 0,0,0\rangle$.

② Circular motion

$\vec{r}(t)$ describes a circle with fixed center $(=(0,0))$ and radius ρ in $\mathbb{R}_{>0}$

We know $|\vec{r}(t)| = \rho$. and we can choose $\boxed{\begin{array}{c}\vec{r}(t) = \langle \rho\cos t, \rho\,\text{sen}\,t\rangle \\ 0\leq t\leq 2\pi\end{array}}$

Recall (Lecture VI): $\underset{\text{(in this case)}}{0 = }\frac{d}{dt}|\vec{r}(t)| = \frac{\vec{r}(t)\cdot\vec{r}\,'(t)}{|\vec{r}(t)|}$ $\equiv \rho$ in this case



We conclude $\vec{r}(t) \perp \vec{r}\,'(t)$ for all values of t.

- $\vec{r}\,'(t) = \langle -\rho\sin t, \rho\cos t\rangle$ Again $|\vec{r}\,'(t)| = \rho$ so $\vec{r}\,'(t) \perp \vec{r}\,''(t)$
- $\vec{r}\,''(t) = \langle -\rho\cos t, -\rho\,\text{sen}\,t\rangle = -\vec{r}(t)$ Again $|\vec{r}\,''(t)| = \rho$.

Theorem (Motion with constant $|\vec{r}_{(t)}|$)

If $\vec{r}$ describes a path on which $|\vec{r}|$ is constant (motion on a circle or sphere centered at the origin), then $\vec{r}_{(t)} \cdot \vec{r}\,'_{(t)} = 0$.

Proof: We use $|\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t)$ & $|\vec{r}_{(t)}|^2 = \text{constant}$.

We differentiate with respect to t:

- $\frac{d}{dt}\Big(\underbrace{|\vec{r}(t)|^2}_{\mathbb{R}\text{-valued function}} \Big) = \frac{d}{dt}(\text{constant}) = 0.$
- $\frac{d}{dt}\big(\vec{r}(t)\cdot\vec{r}(t)\big) \underset{\substack{\downarrow \\ \text{dot product}}}{=} \vec{r}\,'_{(t)} \cdot \vec{r}(t) + \vec{r}_{(t)} \cdot \vec{r}\,'_{(t)} = 2\,\vec{r}_{(t)} \cdot \vec{r}\,'_{(t)}$

Conclusion $0 = 2\,\vec{r}\,'_{(t)} \overset{\text{rule}}{\cdot} \vec{r}_{(t)}$, so $\vec{r}_{(t)} \cdot \vec{r}\,'_{(t)} = 0$ as we wanted to show.

§ 3 Newton's laws of motion:

I. A particle in a state of rest of motion will continue to be so unless an external force is applied to it.

II. $\boxed{m \cdot \vec{a} = \vec{F}}$

m = mass (scalar)

$\vec{a} = \vec{r}\,''$ = acceleration (vector)

$\vec{F}$ = (sum of all) force (vector)

III. To every action, there is an equal and opposite reaction

Gravity force induces acceleration $\boxed{g \approx 9.8\ \frac{m}{s^2} = 32\ ft/s^2}$

Next Time: discuss motions in a gravitational field & with other forces (eg wind, spin, ...)