Recall: II Newton's Law $m \cdot \ddot{a} = F$

Gravity force induces acceleration when $g \approx 9.8 \text{ m/s}^2 = 32 \text{ ft/s}^2$

§1. Two-dimensional motion in a gravitational field (e.g., arrow, baseball, golf ball, ...)

Model for the motion of a projectile in which the only force acting is the gravitational force $\overrightarrow{F} = \langle 0, -mg \rangle = m \overrightarrow{a}(t)$, so $\ddot{a}(t) = \langle 0, -g \rangle$

**GOAL:** Find $\ddot{r}(t)$ such that $\dddot{r}(t) = \overrightarrow{a}(t)$.

$\dddot{r}(t)$ will be unique up to a linear factor $(C_1 + C_2 t)$ (on each component) so to determine $\dddot{r}(t)$ uniquely, we must impose initial conditions.

Initial conditions: can be in 2 forms

1. \( \overrightarrow{r}(0) = \langle x_0, y_0 \rangle \)
2. \( \overrightarrow{v}(0) = \langle v_x, v_y \rangle = \ddot{r}(0) \)

3. \( \overrightarrow{v}(0) = \langle v_x, v_y \rangle \)
4. \( \overrightarrow{v}(0) = \langle v_x, v_y \rangle \)

Typical example: \( \{ x_0 = 0 \} \)

We proceed in 2 steps

**Step 1:** Solve for the velocity, by integrating $\overrightarrow{a}(t)$.

\[ \overrightarrow{v}(t) = \int \overrightarrow{a}(s) \, ds = \int \langle 0, -g \rangle \, ds = \langle 0, -gt \rangle + \mathbf{C} \]

We use the initial conditions to find the constant $\mathbf{C}$; because $\overrightarrow{v}(0) + \mathbf{C} = \mathbf{C}$.
Note: y-component of \( \vec{v}(t) \) decreases linearly with \( t \), so at some time \( t \) it reaches maximal height, namely when

\[-gt_0 + v_0 = 0.\]

Then:

\[ t_0 = \frac{v_0}{g} \quad \text{or} \quad \quad t_0 = \frac{v_0 \sin \alpha}{g} \]

x-component of \( \vec{v}(t) \) is constant > 0 if \( y > 0 \). So, the object always moves forward until it hits the ground.

**STEP 2:** Solve for the position, by integrating \( \vec{v}(t) \).

\[ \vec{r}(t) = \int \vec{v}(s) \, ds = \int <v_1, -gs + v_2> \, ds = <v_1t, -\frac{gt^2}{2} + v_2t + \vec{r}_0> \]

Use initial condition to find \( \vec{r}_0 \).

\[ \vec{r}_0 = \vec{r}(0) = <v_0, v_0> \quad \Rightarrow \quad \vec{r}(t) = <v_1t, -\frac{gt^2}{2} + v_2t + v_0> \]

Other info:
- Maximal height = y-component of \( \vec{r}(t_0) \)

\[ \text{Max height} = y_0 + \frac{v_y^2}{2g} = y_0 + \frac{v_0^2 \sin^2 \alpha}{2g} \quad \text{or} \quad \text{Max height} = y_0 + \frac{v_0^2 \sin^2 \alpha}{2g} \]

Time of flight = \( t_1 \) such that the object hits the ground, i.e. y-component

\[
\frac{1}{2} \vec{r}(t_1) = 0 \quad \& \quad \text{we solve for } t_1.
\]

From

\[ -\frac{1}{2} t_1^2 + v_2 t_1 + v_0 = 0 \quad \& \quad \text{solve for } t_1.\]

Use quadratic formula:

\[ t_1 = \frac{-v_2 \pm \sqrt{v_2^2 + 4g^2y_0}}{2g} \]

But \( t_1 > 0 \) (time!) so there's only one meaningful solution

\[ t_1 = \frac{v_2 + \sqrt{v_2^2 + 4g^2y_0}}{2g} \quad \text{or} \quad t_1 = \frac{v_2 - \sqrt{v_2^2 + 4g^2y_0}}{2g} \]

Range = horizontal distance travelled = x-component of \( \vec{r}(t_1) \).

\[ \text{Range} = v_1 t_1 \quad \& \quad \vec{r}_1 = (v_0 \cos \alpha) t_1, \]

**SPECIAL CASE:** If \( y_0 = 0 \), then \( t_1 = \frac{v_2}{g} \quad \text{or} \quad t_1 = \frac{2v_0 \sin \alpha}{g} \quad \text{and} \quad \text{the range} = \frac{2v_0 v_2}{g} \quad \text{or} \quad \text{the range} = \frac{2v_0^2 \sin 2\alpha}{g}. \]
Three-dimensional motion:

**CASE 1:** Only gravitational force $\vec{F} = <0, 0, -mg>$ and $\vec{a} = <0, 0, -g>$

As in 2D, we impose 2 initial conditions to find $\vec{V}(t)$ and $\vec{r}(t)$.

**CASE 2:** Other forces (e.g. crosswinds, spins, etc.) also apply.

Then $\vec{F} = <0, 0, -mg> + \text{other forces}$.

**Example:** A baseball is hit 3 ft above home plate with an initial velocity of $<30, 30, 80>$ ft/s. The spin on the baseball produces a horizontal acceleration of 5 ft/s² in the northward direction.

1. Find the proton vectors $\vec{p}_z \quad t \geq 0$
2. Determine the time of flight and range of the ball.
3. Determine its maximum height. $g = 32 \text{ ft/s}^2$.

$$\vec{a}(t) = <0, 0, -g> + <0, 5, 0> = <0, 5, -32>$$

$$\vec{V}(t) = \int <0, 5, -32> \, dt = <0, 5t, -32t> + \vec{C}_z$$

Thus $\vec{r}(t) = \int <30, 30, 80 \text{ (initial)} + 30, 30, 80 - 32t> \, dt = <30t, 5t^2 + 30t, 80t - 32t^2> + \vec{C}_z$

$$\vec{C}_z = \vec{r}(0) = <0, 0, 3> \Rightarrow \vec{r}(t) = <30t, \frac{5t^2}{2} + 30t, 3 + 80t - 16t^2>$$

**Time of flight:** $t$, such that $3 + 80t - 16t^2 = 0$

$$t = \frac{-80 \pm \sqrt{80^2 + 4 \cdot 3 \cdot 16}}{-2 \cdot 16} = \frac{-80 \pm 10\sqrt{103}}{-32} \Rightarrow \text{only positive solution } t = \frac{10 + \sqrt{103}}{8} \approx 5 \text{ seconds}.$$

Range $\vec{r}(t) = <150, \frac{25}{2}, 0>$ so range = $|\vec{r}(t)| = \frac{52^{2/3}}{2} \approx 260 \text{ ft}$

**Maximum height:** $t_0$ when $y$-comp of $\vec{V}(t_0) = -32t_0 + 80 = 0 \Rightarrow t_0 = \frac{80}{32} = \frac{5}{2}$

**Height** $y$-comp of $\vec{r}(t_0) = 3 + 80 \cdot \frac{5}{2} - 16 \left( \frac{5}{2} \right)^2 = 103 \text{ ft}$