Lecture VIII (1/29/16) §12.7 (cont.) Motion in space

Recall : II Newton's Law $\boxed{m \cdot \vec{a} = \vec{F}}$

m = mass (scalar)
$\vec{a} = \vec{r}''$ = acceleration (vector)
$\vec{F}$ = (sum of all) force (vector)

gravity force induces acceleration where $\boxed{g \approx 9.8 \frac{m}{s^2} = 32 \frac{ft}{s^2}}$

§1. Two-dimensional motion in a gravitational field (eg arrow, baseball, golf ball, ...)

Model for the motion of a projectile in which the ONLY force acting is the gravitational force $\vec{F} = \langle 0, -mg \rangle = m\vec{a}(t)$,

so $\boxed{\vec{a}(t) = \langle 0, -g \rangle}$ $\quad \vec{F} \downarrow$

GOAL : Find $\vec{r}(t)$ such that $\vec{r}''(t) = \vec{a}(t)$.

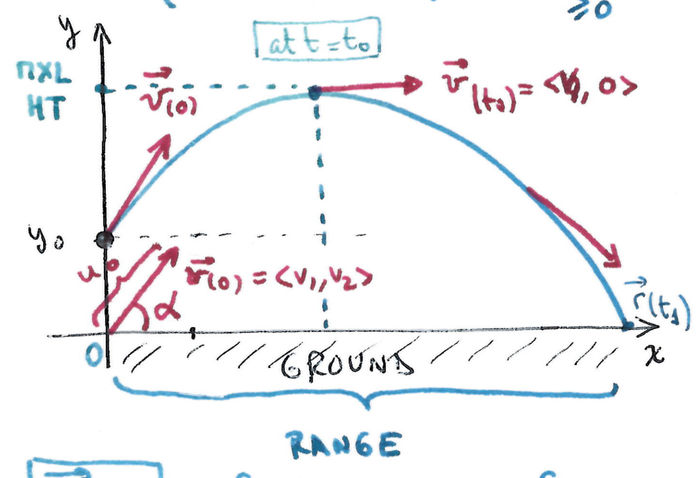
$r(t)$ will be unique up to a linear factor $(\vec{c}_1 + \vec{c}_2 t)$ (vector that is linear on each component)
so to determine $\vec{r}(t)$ uniquely, we must impose 2 initial conditions.

• Initial conditions : come in 2 flavors

(A) • initial position at $t=0$ $\quad \vec{r}(0) = \langle x_0, y_0 \rangle$
• initial velocity at $t=0$ $\quad \vec{v}(0) = \langle v_1, v_2 \rangle = \vec{r}'(0)$

or

(B) • initial position at $t=0$ $\quad \vec{r}(0) = \langle x_0, y_0 \rangle$
• initial speed $v_0 \geq 0$ and angle α $(0 \leq \alpha \leq \pi/2)$, so $\vec{v}(0) = \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle$



Typical example = $\begin{cases} x_0 = 0 \\ y \geq 0 \end{cases}$

We proceed in 2 steps

STEP 1 : Solve for the velocity, by integrating $\vec{a}(t)$.

$$\boxed{\vec{v}(t)} = \int \vec{a}(s)\, ds = \int \langle 0, -g \rangle ds = \boxed{\langle 0, -gt \rangle + \vec{C}}$$

We use the initial conditions to find the constant $\vec{C} = \langle c_1, c_2 \rangle$; because $\vec{v}(0) = \cancel{0} + \vec{C} = \vec{C}$.

Ⓐ $\vec{C} = \langle v_1, v_2 \rangle$ and Ⓑ $\vec{C} = \langle u_0 \cos\alpha, u_0 \sin\alpha \rangle$ 2

Note: • y-component of $\vec{v}(t)$ decreases linearly with t, so at some time t_0 it reaches maximal height, namely when $\boxed{-gt_0 + C_2 = 0}$. Then:

Ⓐ $t_0 = \frac{v_2}{g}$ or Ⓑ $t_0 = \frac{u_0 \sin\alpha}{g}$

• x-component of $\vec{v}(t)$ is constant >0 if $v_1>0$. So, the object always moves forward until it hits the ground.

STEP 2: Solve for the position, by integrating $\vec{v}(t)$.

Ⓐ $\vec{r}(t) = \int \vec{v}(s)\,ds = \int \langle v_1, -gs + v_2 \rangle\,ds = \langle v_1 t, -g\frac{t^2}{2} + v_2 t \rangle + \vec{C}_2$.

Use initial conditions to find $\vec{C}_2$.

$\vec{C}_2 = \vec{r}(0) = \langle 0, y_0 \rangle \Longrightarrow \boxed{\vec{r}(t) = \langle v_1 t, -\frac{gt^2}{2} + v_2 t + y_0 \rangle}$

Ⓑ $\boxed{\vec{r}(t) = \langle (u_0 \cos\alpha) t, -\frac{gt^2}{2} + (u_0 \sin\alpha) t + y_0 \rangle}$

Other info? • Maximal height = y-component of $\vec{r}(t_0)$

Ⓐ $\text{MAX HT} = y_0 + \frac{v_2^2}{g} - \frac{1}{2}\frac{v_2^2}{g} = y_0 + \frac{v_2^2}{2g}$ or Ⓑ $\text{MAX HT} = y_0 + \frac{u_0^2 \sin^2\alpha}{2g}$.

• Time of flight = t_1 such that the object hits the ground, so y-component of $\vec{r}(t_1) = 0$ & we solve for t_1.

Q: How to find t_1?

For Ⓐ $-\frac{g}{2} t_1^2 + v_2 t_1 + y_0 = 0$ & solve for t_1.

Use quadratic formula: $t_1 = \frac{-v_2 \pm \sqrt{v_2^2 + 4\frac{g}{2} y_0}}{2(-g/2)} = \frac{v_2 \mp \underbrace{\sqrt{v_2^2 + 2gy_0}}_{\geq v_2}}{g}$

But $t_1 \geq 0$ (time!) so there's only one meaningful solution

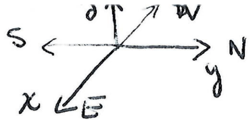
Ⓐ $\boxed{t_1 = \frac{v_2 + \sqrt{v_2^2 + 2gy_0}}{g}}$, Ⓑ $\boxed{t_1 = \frac{u_0 \sin\alpha \pm \sqrt{u_0^2 \sin^2\alpha + 2gy_0}}{g}}$

• Range = horizontal distance travelled = x-component of $\vec{r}(t_1)$.

Ⓐ range $= v_1 t_1$, Ⓑ $= (u_0 \cos\alpha) t_1$

SPECIAL CASE: If $y_0 = 0$, then $t_1 = \frac{2v_2}{g}$ (or $t_1 = \frac{2u_0 \sin\alpha}{g}$) and the range $= \frac{2v_1 v_2}{g}$ (or $= \frac{2u_0^2 \sin\alpha \cos\alpha}{g} = (u_0^2 \sin 2\alpha)/g$)

② Three-dimensional motion:



CASE 1: Only gravitational force $\vec{F} = \langle 0, 0, -mg\rangle$ so $\vec{a}_{(t)} = \langle 0, 0, -g\rangle$

As in 2D, we impose 2 initial conditions to find $\vec{v}_{(t)}$ & $\vec{r}_{(t)}$.

CASE 2: Other forces (eg crosswinds, spins, etc) also apply.

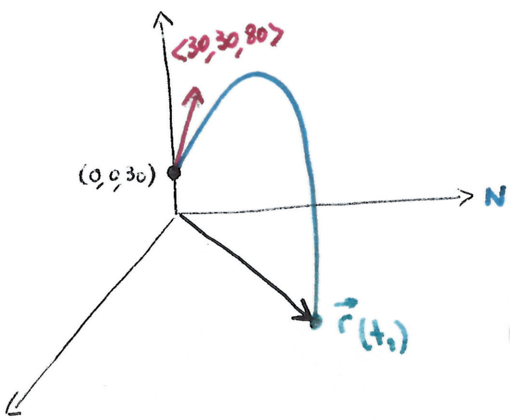
Then $\vec{F} = \langle 0, 0, -mg\rangle +$ other forces.

EXAMPLE: A baseball is hit 3 ft above home plate with an initial velocity of $\langle 30, 30, 80\rangle$ ft/s. The spin on the baseball produces a horizontal acceleration of the ball of 5 ft/s² in the northward direction.

(1) Find the position vectors for $t \geq 0$

(2) Determine the time of flight and range of the ball.

(3) Determine its maximum height. $g = 32$ ft/s².



(1) $\vec{a}_{(t)} = \langle 0, 0, -g\rangle + \langle 0, 5, 0\rangle$
$= \langle 0, 5, -32\rangle$

$\vec{v}_{(t)} = \int \langle 0, 5, -32\rangle \, dt = \langle 0, 5t, -32t\rangle + \vec{C}_1$.

$\vec{C}_1 = \vec{v}_{(0)} = \langle 30, 30, 80\rangle$

$\vec{v}_{(t)} = \langle 30, 5t + 30, 80 - 32t\rangle$

Then $\vec{r}_{(t)} = \int \langle 30, 5s + 30, 80 - 32s\rangle \, ds = \langle 30t, \frac{5t^2}{2} + 30t, 80t - \frac{32t^2}{2}\rangle + \vec{C}_2$

$\vec{C}_2 = \vec{r}_{(0)} = \langle 0, 0, 3\rangle \Rightarrow \boxed{\vec{r}_{(t)} = \langle 30t, \frac{5}{2}t^2 + 30t, 3 + 80t - 16t^2\rangle}$

(2) Time of flight: t_1 such that $3 + 80t_1 - 16t_1^2 = 0$

quadratic formula $= \frac{-80 \pm \sqrt{80^2 + 4 \cdot 3 \cdot 16}}{-2 \cdot 16} = \frac{+10 \pm \sqrt{103}}{4}$ ⇝ only positive solution $t_1 = \frac{10 + \sqrt{103}}{4}$

⇒ Approx 5 seconds. $\boxed{t \approx 5 \text{ sec}}$

Range $\vec{r}(t_1) \approx \langle 150, \frac{425}{2}, 0\rangle$ so range $= |\vec{r}(t_1)| = \frac{5^2\sqrt{433}}{2} \approx \boxed{260 \text{ ft}}$

(3) Maximum height: t_0 when z-comp of $v_{(t_0)} = -32t_0 + 80 = 0 \Rightarrow t_0 = \frac{80}{32} = \frac{5}{2}$

Height z-comp of $\vec{r}_{(t_0)} = 3 + 80 \cdot \frac{5}{2} - 16\left(\frac{5}{2}\right)^2 = \boxed{103 \text{ ft}}$