Lecture IX (2/1/16) §12.8: Lengths of curves

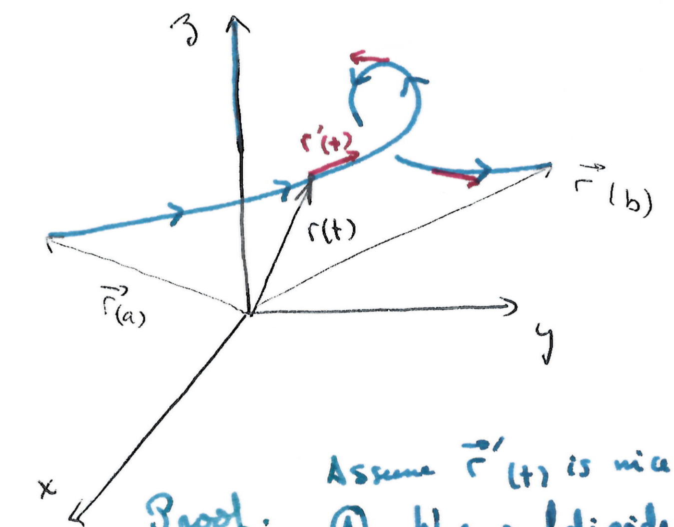
Last time: modelled trajectories of objects in space moving by the action of some forces (eg gravity, wind, spins, etc), with 2 initial conditions: $\vec{r}(t)$ & $r'(t)$

We learned how to compute:
- maximal height achieved & time t_0 at which it's achieved
- time of flight & range travelled

Missing information: how far does the object travel along its flight path?

§1 Arc length

Fix $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ $\quad a \le t \le b \quad$ a parametric curve



Def: The length of the parametric curve is given by

$$\boxed{L = \int_a^b \underbrace{|\vec{r}\,'(t)|}_{=\text{speed!}}\, dt}$$

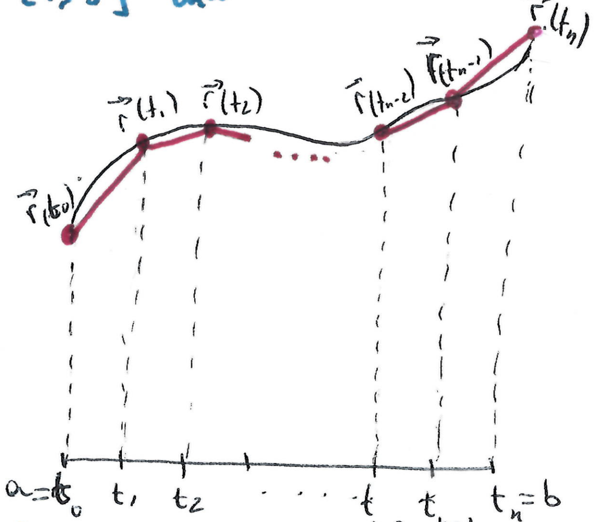
(analog of speed × elapsed time formula)

Assume $\vec{r}\,'(t)$ is nice & smooth.

Proof: ① We subdivide the interval $[a,b]$ into n subintervals of length $\frac{b-a}{n}$, marked by $n+1$ pts:

$$a = t_0 < t_1 < \dots < t_n = b$$

② We draw the polygonal path joining $\vec{r}(t_0), \vec{r}(t_1), \dots, \vec{r}(t_{n-1})$ & $\vec{r}(t_n)$ approximating the curve.



Each segment has length $|\vec{r}(t_j) - \vec{r}(t_{j-1})|$

But (*) $|\vec{r}(t_j) - \vec{r}(t_{j-1})| = |\vec{r}\,'(s_j)(t_j - t_{j-1}) + \varepsilon|$ for some point $t_{j-1} \le s_j \le t_j$

Intermediate value theorem in each component.

And ε can be made very small ($< 1/n^2$) as n grows. & $\boxed{t_j - t_{j-1} = \frac{b-a}{n}}$

So $\left|\vec{r}\,'(s_j)\left(\frac{b-a}{n}\right) + \varepsilon\right| \approx \left|\vec{r}\,'(t_j)\cdot\left(\frac{b-a}{n}\right)\right|$ for n large

By definition: $\displaystyle\int_a^b |\vec{r}\,'(t)|\,dt = \lim_{n\to\infty}\sum_{j=1}^{n} |\vec{r}\,'(t_j)|\left|\frac{b-a}{n}\right| \underset{(*)}{\approx} \lim_{n\to\infty}\sum_{j=1}^{n}|\vec{r}(t_j)-\vec{r}(t_{j-1})| = L$.

where L = length of the curve

Example: $\vec{r}(t) = \langle e^t, e^t\sin t, e^t\cos t\rangle \quad 0\le t\le 4.$

$\vec{r}\,'(t) = \langle e^t, e^t\cos t + e^t\sin t, e^t\cos t - e^t\sin t\rangle$

So $|\vec{r}\,'(t)| = \sqrt{(e^t)^2 + (e^t)^2(\cos t+\sin t)^2 + e^t(\cos t - \sin t)^2}$

$= e^t\sqrt{1 + \sin^2 t + \cos^2 t + 2\sin t\cos t + \cos^2 t + \sin^2 t - 2\sin t\cos t}$

$= e^t\sqrt{3}$

So Length of the curve $= \displaystyle\int_0^4 \sqrt{3}e^t\,dt = \sqrt{3}(e^4-1).$

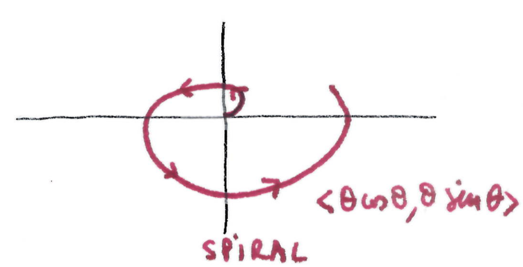
Note: Often, it is very hard to find the antiderivative of $|\vec{r}\,'(t)|$ (eg, trajectories of planetary orbits $\vec{r}(t) = (a\cos t, b\sin t)$ $a\ne b$.) In these cases, numerical approximation methods are used to compute L.

§2. Arc length under changes of coordinates

Recall (POLAR coords) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\alpha\le\theta\le\beta.$ We assume $r = f(\theta)$ (eg: $r=\theta$ as in the spiral)

So $\vec{R}(\theta) = \langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle$ ~~gives~~ $R'(\theta) = \langle f'(\theta)\cos\theta + f(\theta)(-\sin\theta), f'(\theta)\sin\theta + f(\theta)\cos\theta\rangle$

$\Rightarrow |R'(\theta)| = \sqrt{f'(\theta)^2 + f(\theta)^2}$

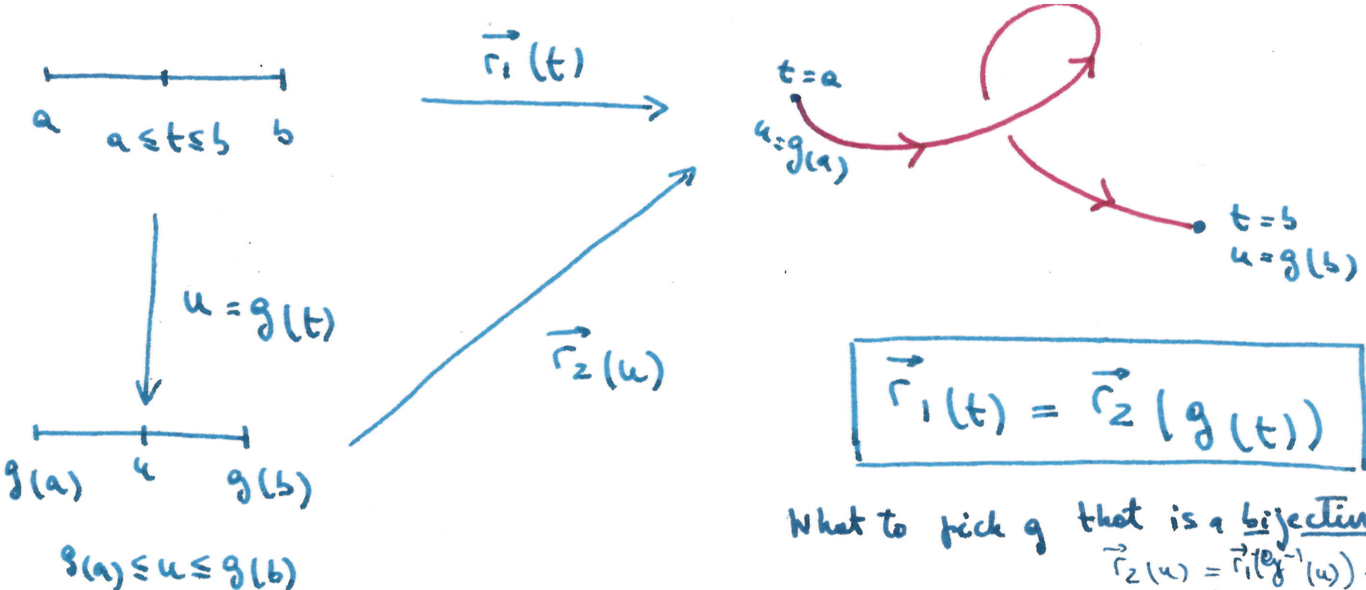


Conclusion: $L = \displaystyle\int_\alpha^\beta \sqrt{f'^2(\theta) + f^2(\theta)}\,d\theta.$

Q: What about other reparameterizations?

How does L change? It shouldn't!

How to choose the parameterization that gives an easy way of calculating L?



$$\boxed{\vec{r}_1(t) = \vec{r}_2(g(t))}$$

What to pick g that is a bijection (1-to-1 map)
$\vec{r}_2(u) = \vec{r}_1(g^{-1}(u))$.

Using this approach, we get several parameterizations of the same curve C.

Examples ⓪ $\vec{r}_1(t) = \langle 1-t, t, t^2 \rangle$ $\quad 0 \le t \le 1$

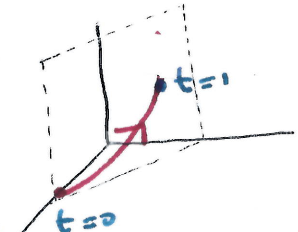
① $u = e^t$ $\quad 1 \le u \le e$ $\quad$ (g = exponential is increasing function)
Then: $t = \ln u$

$\vec{r}_2(u) = \vec{r}_1(\ln u) = \langle 1 - \ln u, \ln u, (\ln u)^2 \rangle$ $\quad 1 \le u \le e$

② $u = \sin^{-1} t$ $\quad 0 \le u \le \frac{\pi}{2}$ $\quad$ ($t = \sin u$) is increasing on $[0,1]$.

$\vec{r}_3(u) = \vec{r}_1(\sin u) = \langle 1 - \sin u, \sin u, \sin^2 u \rangle$ $\quad 0 \le u \le \frac{\pi}{2}$

All 3 functions parameterize the same curve:
a parabola in the plane $y + x = 1$.

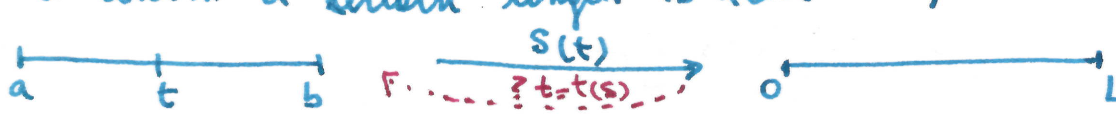


• Most natural parameterization: by arc length! $\vec{r}_2(s)$ such that length travelled at time s is exactly s.

Def: $s(t)$ = length of the curve from a to t

$$\boxed{s(t) = \int_a^t |\vec{r}\,'(\tau)|\, d\tau}$$

Note: $s(t)$ is strictly increasing (hence 1-to-1 map), so in principle we should be able to express t as a function of the length (find the time t at which a certain length is achieved). Then: $t = t(s)$ and $\vec{r}_1(s) = \vec{r}(t(s))$ where s is the arc length parameter



Example: ① $\vec{r}(t) = \langle e^t, e^t \sin t, e^t \cos t\rangle$ $\quad 0 \le t \le 4.$

Earlier today: $|\vec{r}'(t)| = \sqrt{3}\, e^t$, so $s(t) = \int_0^t \sqrt{3}\, e^u\, du = \sqrt{3}(e^t - 1)$

$L = \sqrt{3}(e^4 - 1)$

So to get t from s: $\quad e^t = 1 + \frac{s}{\sqrt{3}} \Rightarrow t = \ln\left(1 + \frac{s}{\sqrt{3}}\right) \quad 0 \le s \le L$

We conclude: $\vec{r}_1(s) = \vec{r}(t(s))$

$$= \left\langle \left(\frac{s}{\sqrt{3}} + 1\right), \left(\frac{s}{\sqrt{3}} + 1\right) \sin\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \left(\frac{s}{\sqrt{3}} + 1\right) \cos\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right)\right\rangle$$

Note: $\vec{r}_1(s)$ = position vector of the point in the curve at a distance s from $\vec{r}(0) = \langle 1, 0, 1\rangle$ for $0 \le s \le \sqrt{3}(e^4 - 1)$.

② $\vec{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3\rangle \quad 0 \le t \le 1$

$\vec{r}'(t) = \langle 2, 2t, t^2\rangle \Rightarrow |\vec{r}'(t)| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2.$

So $s(t) = \int_0^t (u^2 + 2)\, du = \left.\frac{u^3}{3} + 2u\right|_{u=0}^{u=t} = \frac{t^3}{3} + 2t$

So $L = s(1) = \frac{1}{3} + 2 = \frac{7}{3}$

Here, we can't find a nice way to rewrite t as a function of s. (equation is a cubic).

③ $\vec{r}(t) = \langle 3\sin t, 4t, 3\cos t\rangle \quad 0 \le t \le 5$

$\vec{r}'(t) = \langle 3\cos t, 4, -3\sin t\rangle \Rightarrow |\vec{r}'(t)| = \sqrt{9 + 16} = 5$

So $s(t) = \int_0^t 5\, dt = 5t \Rightarrow t = \frac{s}{5}.$

$\vec{r}_1(s) = \vec{r}\left(\frac{s}{5}\right) = \left\langle 3\sin\left(\frac{s}{5}\right), \frac{4}{5}s, 3\cos\left(\frac{s}{5}\right)\right\rangle \quad 0 \le s \le 25$

Position vector of the point on the curve at a distance 5 from $\langle 0, 0, 1\rangle = \vec{r}(0)$

$= \vec{r}_1(5) = \langle 3\sin(1), 4, 3\cos(1)\rangle.$

Theorem: If $|\vec{r}'(t)| = 1$ for all $a \le t \le b$, then $s(t) = \int_a^t 1 = t - a$ and the parameter t corresponds to arc length.