Lecture IX (2/1/16) §12.8: Lengths of curves

Last time: modelling trajectories of objects in space acting by the action of sum forces (e.g. gravity, wind, spins, etc., with 2 initial conditions \( \vec{r}(0), \vec{r}'(0) \))

We learned how to compute:

1. maximal height achieved & time to, at which it's achieved
2. time of flight & range travelled

Missing information: how far does the object travel along its flight path?

§1 Arc length

Fix \( \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \) \( a \leq t \leq b \) a parametric curve

**Def.** The length of the parametric curve is given by

\[
L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt
\]

(Analog of speed = elapsed time formula)

**Proof:**
1. We subdivide the interval \([a, b]\) into \(n\) subintervals of length \(b-a\) \(n\), marked by \(n+1\) pts:
   \[a = t_0 < t_1 < \cdots < t_n = b\]
2. We draw the polygonal path joining \(\vec{r}(t_0), \vec{r}(t_1), \ldots, \vec{r}(t_n)\)
   approximating the curve.

   Each segment has length \(\left| \vec{r}(t_j) - \vec{r}(t_{j-1}) \right|\)

   But \(\left| \vec{r}(t_j) - \vec{r}(t_{j-1}) \right| = \left| \vec{r}'(s_j)(t_j - t_{j-1}) + \epsilon_j \right| \quad \text{for some point} \)

   \[t_{j-1} \leq s_j \leq t_j\]

   And \(\epsilon\) can be made very small as \(n\) grows. \& \[
   \left| t_j - t_{j-1} \right| = \frac{b-a}{n}
   \]
So \( |\vec{r}'(s_j) \cdot (b-c) + \epsilon| \approx |\vec{r}'(t_j) \cdot (b-c)| \) for \( n \to \infty \).

By definition, \( \int_a^b |\vec{r}'(t)| \, dt = \lim_{n \to \infty} \sum_{j=1}^n |\vec{r}'(t_j)| \cdot \epsilon_{j-1} \approx \lim_{n \to \infty} \sum_{j=1}^n |\vec{r}(t_j) - \vec{r}(t_{j-1})| = L. \)

where \( L = \text{length of the curve} \)

\( \text{(x)} \)

Example: \( \vec{r}(t) = <e^t, e^t \cos t, e^t \sin t> \) \( \alpha \leq t \leq \beta. \)

\( \vec{r}'(t) = <e^t, e^t \cos t + e^t \sin t, e^t \cos t - e^t \sin t> \)

So \( |\vec{r}'(t)| = \sqrt{(e^t)^2 + (e^t \cos t + e^t \sin t)^2 + (e^t \cos t - e^t \sin t)^2} \)

\( = e^t \sqrt{1 + \sin^2 t + 2 \sin t \cos t + \cos^2 t + \sin^2 t - 2 \sin t \cos t} \)

\( = e^t \sqrt{3} \)

So the length of the curve \( L = \int_0^\beta e^t \, dt = \sqrt{3} (e^\beta - 1). \)

Note: Often, it is very hard to find the antiderivative of \( |\vec{r}'(t)| \).

(eg, trajectories of planetary orbits \( \vec{r}(t) = \langle a \cos t, b \sin t \rangle \quad a \neq b. \))

In these cases, numerical approximation methods are used to compute it.

\[ \text{§2. Arc length under changes of coordinates} \]

Recall \( \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad 0 \leq \theta \leq \beta. \)

We assume \( r = f(\theta) \)

(eg: \( r = \theta \) in the spiral)

So \( \vec{R}(\theta) = \langle f(\theta) \cos \theta, f(\theta) \sin \theta \rangle \quad \text{gives} \quad R'(\theta) = \langle f'(\theta) \cos \theta + f(\theta) \sin \theta, f'(\theta) \sin \theta + f(\theta) \cos \theta \rangle \)

\( \Rightarrow |R'(\theta)| = \sqrt{f'(\theta)^2 + f(\theta)^2} \)

Conclusion: \( L = \int \sqrt{f'(\theta)^2 + f(\theta)^2} \, d\theta. \)

Q: What about other reparameterizations?

How does \( L \) change? It shouldn't!

How to choose the parameterization that gives an easy way of calculating \( L \)?
Using this approach, we get several parameterizations of the same curve C.

Examples:

1. \( \textbf{r}_1(t) = \langle 1-t, t, t^2 \rangle \quad \text{over} \quad t \in [0,1] \)

2. \( u = e^t \quad 1 \leq u \leq e \) (\( g = \text{exponential is increasing function} \))

   Then: \( t = \ln u \)

   \( \textbf{r}_2(u) = \textbf{r}_1(\ln u) = \langle 1-\ln u, \ln u, (\ln u)^2 \rangle \quad 1 \leq u \leq e \)

3. \( u = \sin^2 t \quad 0 \leq u \leq \frac{\pi}{2} \) (\( t = \sin u \)) is increasing on \([0,1]\).

   \( \textbf{r}_3(u) = \textbf{r}_1(\sin u) = \langle 1-\sin u, \sin u, \sin^2 u \rangle \quad 0 \leq u \leq \frac{\pi}{2} \)

All 3 functions parameterize the same curve: a parabola in the plane \( y+x=1 \).

Most natural parameterization: by arc length!

Def: \( S(t) = \text{length of the curve from } a \text{ to } t \)

\[ S(t) = \int_a^t |\textbf{r}'(u)| \, du \]

Note: \( S(t) \) is strictly increasing, so in principle we should be able to express \( t \) as a function of the length \( S \) (find the time \( t \) at which a certain length is achieved). Then \( t = t(S) \) and \( \textbf{r}(t(S)) = \textbf{r}(S) \), where \( S \) is the arc length parameter.

\[ a \rightarrow t \rightarrow b \quad \text{with } \int_a^b \frac{1}{S(t)} \, dt = L = \text{length of the curve} \]
Example: \( \vec{r}(t) = \langle e^t, e^{t \sin t}, e^{t \cos t} \rangle \quad 0 \leq t \leq 4 \).

Earlier today: \( \vec{r}'(t) = \vec{r}'(t) + \frac{t}{\sqrt{3}} e^t \), so \( \vec{S}(t) = \int \vec{r}'(t) \, dt = \int \frac{t}{\sqrt{3}} e^t \, dt = \frac{t}{\sqrt{3}}(e^t - 1) \)

So to get \( t \) from \( s \): \( e^t = 1 + \frac{s}{\sqrt{3}} \Rightarrow \ln(1 + \frac{s}{\sqrt{3}}) \quad 0 \leq s \leq 7 \)

We conclude: \( \vec{r}_1(s) = \vec{r}_1(t(s)) \)

\[ \begin{align*} &\vec{r}_1(s) = \langle \frac{(s + 1)}{\sqrt{3}}, \frac{(s + 1)}{\sqrt{3}} \sin(\ln(\frac{s + 1}{\sqrt{3}})), \frac{(s + 1)}{\sqrt{3}} \cos(\ln(\frac{s + 1}{\sqrt{3}})) \rangle \\ &\text{Note: } \vec{r}_1(s) \text{ is position vector of the point on the curve at a distance } s \text{ from } \vec{r}_1(0) = \langle 1, 0, 1 \rangle. \end{align*} \]

(2) \( \vec{r}(t) = \langle 2t, t^2, \frac{1}{3} t^3 \rangle \quad 0 \leq t \leq 1 \)

\[ \vec{r}'(t) = \langle 2, 2t, t^2 \rangle \Rightarrow |\vec{r}'(t)| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2 \]

So \( S(t) = \int_0^t (u^2 + 2) \, du = \frac{u^3 + 2u}{3} \bigg|_{u=0}^{u=t} = \frac{t^3}{3} + 2t \)

So \( L = S(1) = \frac{1}{3} + 2 = \frac{7}{3} \) \quad Here, we can't find a way to rewrite \( t \) as a function of \( s \).

(3) \( \vec{r}(t) = \langle 3 \sin t, 4t, 3 \cos t \rangle \quad 0 \leq t \leq 5 \)

\[ \vec{r}'(t) = \langle 3 \cos t, 4, -3 \sin t \rangle \Rightarrow |\vec{r}'(t)| = \sqrt{9 + 16} = 5 \]

So \( S(t) = \int_0^t 5 \, dt = 5t \Rightarrow t = \frac{s}{5} \)

\[ \vec{r}_1(s) = \vec{r}_1(\frac{s}{5}) = \langle 3 \sin(\frac{s}{5}), 4 \frac{s}{5}, 3 \cos(\frac{s}{5}) \rangle \quad 0 \leq s \leq 25 \]

Position vector of the point on the curve at a distance \( s \) from \( \vec{r}(0) = \vec{r}_1(0) \)

\[ \vec{r}_1(5) = \langle 3 \sin(1), 4, 3 \cos(1) \rangle \]

Theorem: If \( |\vec{r}'(t)| = 1 \) for all \( a \leq t \leq b \), then \( S(t) = \int_a^t = t - a \)

and the parameter \( t \) corresponds to arc length.