Lecture X (2/3/16) § 12.9: Curvature & normal vectors

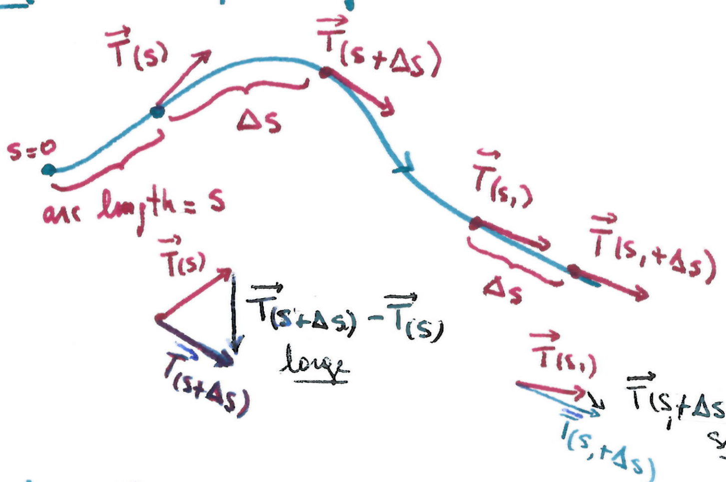
§ 1 Curvature:

Fix $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ for $a \le t \le b$.

Measures how fast a curve turns at a point (= scalar!).
Same as measuring how fast the tangent vectors turn, or even better, how fast the unit tangent vectors turn (magnitude doesn't matter, only the directions matter).

Recall: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ unit tangent vector to $\vec{r}(t)$ at t.

Q: How quickly does T change in direction as we move along the curve?



$s = s(t)$ arc length at time t.
The change for $\vec{T}$ is measured by small increments in arc length, so it's better to assume $\vec{r}$ is parameterized by arc length
ie length $(\vec{r}(s)) = s$.

Def: The curvature is defined as $\left| \dfrac{d}{ds} \vec{T} \right|$ and denoted by $\kappa(s)$.

Note:
- $\kappa(s) \ge 0$ for every $s \ge 0$
- Large value of $\kappa(s)$ indicates a tight curve that changes direction quickly at s.
- A small value of $\kappa(s)$ indicates the curve is relatively flat, and changes directions slowly at s. In particular $\kappa(s) \equiv 0$ if and only if $\vec{r}$ is a line.

Recall: Parameterization by arc length is natural but very hard to calculate.

Better formula for κ? Use chain rule! $s = s(t)$ (arc length is a function of time).

$\vec{r}(t) = \vec{r}_1(s(t))$ (→ arc length param.) so $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{r}_1\,'(s(t)) \cdot s'(t)}{|\vec{r}_1\,'(s(t))|\,|s'(t)|} = \vec{T}_1(\vec{s}(t))$

$\dfrac{s'(t)}{|s'(t)|} = 1$ (s increasing)

$\vec{T}(t) = \vec{T}_1(s(t))$ so $\dfrac{d\vec{T}}{dt} = \dfrac{d\vec{T}_1}{ds} \cdot \dfrac{ds}{dt}$

$s(t) = \displaystyle\int_a^t |\vec{r}\,'(u)|\, du$ so $\boxed{\dfrac{ds}{dt} = |\vec{r}\,'(t)| = |\vec{v}(t)|}$

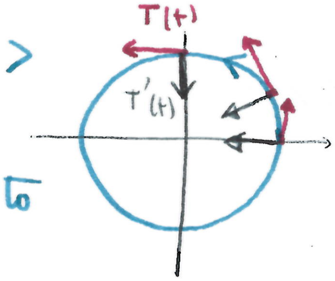
Theorem 1: $\kappa(t) = \dfrac{1}{|\vec{v}(t)|} \cdot \left|\dfrac{d\vec{T}}{dt}\right| = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$ (for any parametrization)

Example $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$ $(0 \le t \le 2\pi)$ $a \ge 0$: [circle of radius $a \ge 0$]

$\vec{r}'(t) = \langle -a\sin t, a\cos t\rangle \Rightarrow |\vec{r}'(t)| = a$

$\vec{T}(t) = \langle -\sin t, \cos t\rangle \Rightarrow \vec{T}'(t) = \langle -\cos t, -\sin t\rangle$

$|\vec{T}'(t)| = 1.$



$\Rightarrow \kappa(t) = \dfrac{1}{a}$ is constant and inverse proportional to the radius a.

(The larger the radius, the smaller the curvature!)

§2 Alternative curvature formula in $\mathbb{R}^3$:

Theorem 2 $\kappa(t) = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$.

Proof: $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ and $|\vec{r}'(t)| = \dfrac{ds}{dt}$, so $\vec{r}'(t) = \dfrac{ds}{dt}\vec{T}(t)$

Differentiate both sides: & use product rule
$$\boxed{\vec{r}''(t) = \underbrace{\frac{d^2s}{d^2t}}_{= s''(t)}\vec{T}(t) + \frac{ds}{dt}\vec{T}'(t)} \quad (*)$$

Then $\vec{r}'(t) \times \vec{r}''(t) = \dfrac{ds}{dt}\vec{T}(t) \times \left(\dfrac{d^2s}{d^2t}\vec{T}(t) + \dfrac{ds}{dt}\vec{T}'(t)\right)$

$\underset{\text{properties of } \times}{=} \dfrac{ds}{dt}\dfrac{d^2s}{d^2t}\underbrace{\left(\vec{T}(t) \times \vec{T}(t)\right)}_{=0} + \left(\dfrac{ds}{dt}\right)^2 \vec{T}(t) \times \vec{T}'(t)$

But $|\vec{T}(t)| = 1$ for all t, so we know $\vec{T}'(t) \perp \vec{T}(t)$, so $\theta = \frac{\pi}{2}$

$|\vec{T}(t) \times \vec{T}'(t)| = |\vec{T}(t)|\,|\vec{T}'(t)| \sin(\theta) = |\vec{T}|\,|\vec{T}'| = |\vec{T}'|$

Then $|\vec{r}'(t) \times \vec{r}''(t)| = \left|\dfrac{ds}{dt}\right|^2 |\vec{T}'| \underset{\text{def.}}{=} |\vec{r}'(t)|^2 |\vec{T}'| \underset{\text{Thm 1}}{=} |\vec{r}'(t)|^3 \kappa(t)$ □

Example (above): $\vec{r}''(t) = \langle -a\cos t, -a\sin t, 0\rangle$

$|\vec{r}'(t) \times \vec{r}''(t)| = a \cdot a \left|\begin{vmatrix} i & j & k \\ -\sin t & \cos t & 0 \\ \cos t & \sin t & 0 \end{vmatrix}\right| = a^2 \,|(\sin^2 t + \cos^2 t)\,k| = a^2$

$\Rightarrow \kappa(t) = \dfrac{a^2}{a^3} = \dfrac{1}{a}$ ✓.

§3. Principal Unit Normal Vector

GOAL: Determine the direction in which the curve turns at a given time t.

Equivalently: determine the rate of change of the tangent directions, and record it's direction, not the magnitud changes.

Again, we assume parameterization by arc length ($\vec r(s)$) and $\kappa(s) \neq 0$ for all s.

Def: $\vec N(s) = \dfrac{\frac{d\vec T}{ds}}{\left|\frac{d\vec T}{ds}\right|} = \dfrac{1}{\kappa(s)}\dfrac{d\vec T}{ds}$ is the principal unit normal vector

As with curvature, we use the chain rule to get a formula for any parametric curve: [$t=t(s)$, $\vec T_{(s)}(t) = \vec T_1(s)$ and $\vec r_1(s) = \vec r(t(s))$] ↳ arc length param.

$$\vec N(t) = \frac{\frac{d\vec T}{dt}\cdot\frac{dt}{ds}}{\left|\frac{d\vec T}{dt}\right|\left|\frac{dt}{ds}\right|} = \frac{\frac{d\vec T}{dt}}{\left|\frac{d\vec T}{dt}\right|}$$

because t is an increasing function so $\frac{dt}{ds} > 0$.

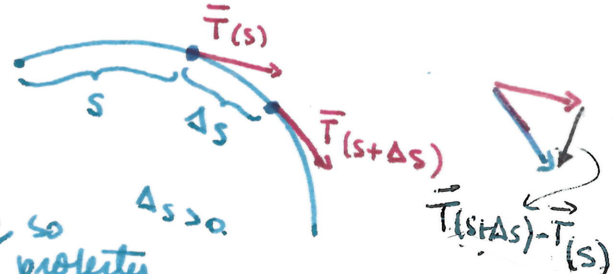
Why this notation? "inner normal"

Properties: (1) $\vec T(t)$ and $\vec N(t)$ are orthogonal for every t. (Proof: on page 2)

(2) $\vec N(t)$ points to the inside of the curve, that is, in the direction in which the curve is turning.

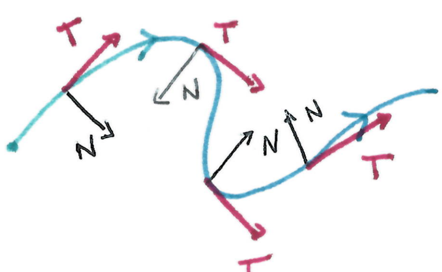
Pf/ $\vec N(t)$ and $\frac{d\vec T(s)}{ds}$ have the same direction ($\kappa(s) > 0$)

$$\frac{d\vec T(s)}{ds} = \lim_{\Delta s \to 0} \frac{\vec T(s+\Delta s) - \vec T(s)}{\Delta s}$$



$\vec T(s+\Delta s) - \vec T(s)$ points inwards when $\Delta s > 0$, so the limit has the same property

(3) $\vec N(t)$ has magnitude 1 ("unit" normal)



§4. Application: components of the acceleration

Def: 1) $\boxed{a_T(t)}$ = scalar component of $\vec a(t)$ along $\vec T(t)$

$$= \frac{\vec a(t)\cdot\vec T(t)}{|\vec T(t)|} = \vec a(t)\cdot\vec T(t) = \boxed{\frac{d^2 s}{d^2 t}}$$

(*) on page 2

$\vec T \cdot \vec T = 1$

$\vec T \cdot \vec T' = 0$

(2) $\boxed{a_N(t)}$ = scalar comp of $\vec a(t)$ along $\vec N(t)$

$$= \vec a(t)\cdot\vec N(t) = \frac{\vec a(t)\cdot\vec T'(t)}{|\vec T'(t)|} \underset{(*)}{=} \frac{ds}{dt}|\vec T'(t)| = |\vec r'(t)||\vec T'(t)| = \boxed{\kappa(t)\,|\vec r'(t)|^2}$$