§1 Curvature:

Fix \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) for \( a \leq t \leq b \).

Measure how fast a curve turns at a point (= scalar!).
Same as measuring how fast the tangent vectors turn, or even better,
how fast the unit tangent vectors turn (magnitude doesn't matter, only the directions matter).
Recall: \( \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \), unit tangent vector to \( \mathbf{r}(t) \) at \( t \).

Q: How quickly does \( \mathbf{T} \) change in direction as we move along the curve?

\[ S = s(t) \]

arc length at time \( t \).

The change \( \Delta \mathbf{T} \), measured by small increments in arc length, so it's better to assume \( \mathbf{r}(t) \) is parameterized by arc length, i.e., length \( (\mathbf{r}'(s)) = s \).

Def: The curvature is defined as \( \left| \frac{d}{ds} \mathbf{T} \right| \) and denoted by \( K(s) \).

Note: \( K(s) > 0 \) for every \( s > 0 \)
- Large value of \( K(s) \) indicates a tight curve that changes direction quickly at \( s \).
- A small value of \( K(s) \) indicates the curve is relatively flat, and changes direction slowly at \( s \). In particular, \( K(s) = 0 \) if and only if \( \mathbf{T} \) is a line.

Recall: Parameterization by arc length is natural but very hard to calculate.

Better formula for \( K \)? Use chain rule! \( S = s(t) \) (arc length is a function of time).

\[ \mathbf{T}(t) = \frac{\mathbf{r}'(s(t))}{\|\mathbf{r}'(s(t))\|} \Rightarrow \frac{\mathbf{T}}{\mathbf{r}(t)} = \frac{\mathbf{r}'(s(t)) s'(t)}{\|\mathbf{r}'(s(t))\|} = \frac{\mathbf{T}(s(t)) s'(t)}{\|\mathbf{r}'(s(t))\|} \]

\[ \frac{d}{dt} \mathbf{T} = \frac{d}{ds} \mathbf{T} \cdot \frac{ds}{dt} \]

\[ S(t) = \int_a^t \mathbf{r}'(u) \, du \Rightarrow \frac{ds}{dt} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left| \mathbf{r}'(t) \right| \]

Theorem: \( K(t) = \frac{\mathbf{T}(t)}{\|\mathbf{T}(t)\|} \).

\[ \frac{d}{dt} \mathbf{T} = \frac{1}{\|\mathbf{T}(t)\|} \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{T}(t)}{\|\mathbf{T}(t)\|^2} \quad \text{(for any parametrization)} \]
Example \( \mathbf{r}(t) = \langle a \cos t, a \sin t, 0 \rangle \quad (0 \leq t \leq 2\pi) \Rightarrow \mathbf{r}'(t) = \langle -a \sin t, a \cos t, 0 \rangle \Rightarrow \mathbf{T}(t) = \langle -a \sin t, a \cos t, 0 \rangle \Rightarrow \mathbf{T}'(t) = \langle a \cos t, -a \sin t, 0 \rangle \Rightarrow |\mathbf{T}'(t)| = 1, |\mathbf{T}''(t)| = 1. \)

\( \Rightarrow K(t) = \frac{1}{a} \) is constant and inversely proportional to the radius, \( a \).

(The larger the radius, the smaller the curvature!)

§2 Alternative curvature formula in \( \mathbb{R}^3 \):

**Theorem** \( K(t) = \frac{\mathbf{T}'(t) \times \mathbf{T}''(t)}{|\mathbf{T}'(t)|^3} \).

**Proof:** \( \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \) and \( |\mathbf{r}'(t)| = \frac{ds}{dt} \), so \( \mathbf{r}'(t) = \frac{ds}{dt} \mathbf{T}(t) \)

Differentiate both sides:

\[ \mathbf{r}''(t) = \frac{d^2s}{dt^2} \mathbf{T}(t) + \frac{ds}{dt} \mathbf{T}'(t) \]

\[ \text{Then } \mathbf{T}'(t) \times \mathbf{T}''(t) = \frac{d^2s}{dt^2} \mathbf{T}(t) \times \mathbf{T}'(t) \]

\[ = \frac{d^2s}{dt^2} (|\mathbf{T}(t)| \mathbf{T}(t) + \frac{ds}{dt} \mathbf{T}'(t)) \]

\[ = \frac{d^2s}{dt^2} \mathbf{T}(t) \times \mathbf{T}'(t) \]

But \( |\mathbf{T}(t)| = 1 \) for all \( t \), so we know \( \mathbf{T}(t) \perp \mathbf{T}'(t) \), so \( \theta = \frac{\pi}{2} \)

\[ |\mathbf{T}(t) \times \mathbf{T}'(t)| = 1 \]

\[ \Rightarrow |\mathbf{T}'(t) \times \mathbf{T}''(t)| = 1 |\mathbf{T}'(t)| |\mathbf{T}''(t)| \sin(\theta) = 1 |\mathbf{T}'||\mathbf{T}''| = |\mathbf{r}'(t)||\mathbf{r}''(t)| = |\mathbf{r}'|^2 |\mathbf{r}''| = |\mathbf{r}'|^3 K(t) \]

**Example (above):** \( \mathbf{r}'(t) = \langle -a \cos t, -a \sin t, 0 \rangle \)

\[ |\mathbf{r}'(t) \times \mathbf{r}''(t)| = a^2 |(\sin^2 t + \cos^2 t)k - \cos t \sin t | = a^2 \]

\[ \Rightarrow K(t) = \frac{a^2}{a^2} = \frac{1}{a} \]
§3. Principal Unit Normal Vector

**Goal:** Determine the direction in which the curve turns at a given point.

Equivalently, determine the rate of change of the tangent direction, and record its direction, not the magnitude changes.

Again, we assume parameterization by arc length \( r(s) \) and \( K(s) \neq 0 \) for all \( s \).

\[
\vec{N}(s) = \frac{\frac{d\vec{T}}{ds}}{\left| \frac{d\vec{T}}{ds} \right|} = \frac{\frac{d\vec{T}}{ds}}{K(s) ds}
\]

As with curvature, we use the chain rule to get a formula for any parameterized curve:

\[
\vec{N}(t) = \frac{\frac{d\vec{T}}{dt} \cdot \frac{dt}{ds}}{\left| \frac{d\vec{T}}{dt} \right|} = \frac{\frac{d\vec{T}}{dt}}{\left| \frac{d\vec{T}}{dt} \right|} \frac{ds}{dt}
\]

Why this notation? "inner normal"

**Properties:**
1. \( \vec{T}(t) \) and \( \vec{N}(t) \) are orthogonal for every \( t \). (Proof: in page 2)
2. \( \vec{N}(t) \) points to the inside of the curve, that is, in the direction in which the curve is turning.
3. \( \vec{N}(t) \) and \( \frac{d\vec{T}}{ds} \) have the same direction \( K(s) > 0 \)

\[
\frac{d\vec{T}}{ds} = \lim_{\Delta s \to 0} \frac{\vec{T}(s+\Delta s) - \vec{T}(s)}{\Delta s}
\]

\( \vec{T}(s+\Delta s) - \vec{T}(s) \) points inward when \( \Delta s > 0 \), so \( \Delta s > 0 \) in the limit has the same property

(3) \( \vec{N}(t) \) has magnitude 1 ("unit" normal)

§4. Application: Components of the acceleration

**Def:** \( a_T(t) = \) component of \( \vec{a}(t) \) along \( \vec{T}(t) \)

\[
a_T(t) = \frac{\vec{a}(t) \cdot \vec{T}(t)}{\left| \vec{T}(t) \right|} = \frac{\left| \frac{d^2\vec{s}}{dt^2} \right| \vec{T}(t)}{\left| \vec{T}(t) \right|} = \frac{d^2\vec{s}}{dt^2} \cdot \vec{T}(t)
\]

\[
2. a_N(t) = \text{scalar comp of } \vec{a}(t) \text{ along } \vec{N}(t) = \frac{\vec{a}(t) \cdot \vec{N}(t)}{\left| \vec{N}(t) \right|} = \frac{\frac{d\vec{T}}{dt} \cdot \vec{T}(t)}{\left| \vec{T}(t) \right|} = \frac{\frac{d}{dt} \left( \frac{d\vec{T}}{dt} \right)}{\left| \vec{T}(t) \right|} = \frac{d^2\vec{s}}{dt^2} \cdot \vec{T}(t)
\]