Lecture XI (2/5/16) §12.9 & 13.1 : Planes and surfaces

Recall: Defined $\vec T(t)$, $\vec N(t)$ & curvature $\kappa(t)$ of a curve $\vec r(t) = \langle f(t), g(t), h(t)\rangle$ for $a \le t \le b$.

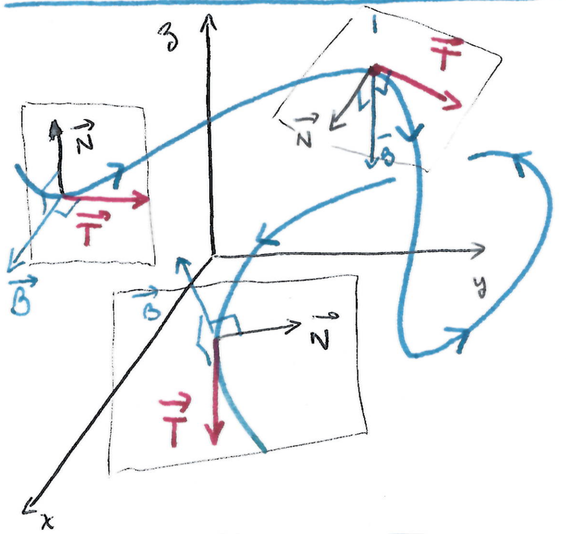
$$\vec N(t) = \frac{\frac{d\vec T}{dt}(t)}{\left|\frac{d\vec T}{dt}(t)\right|}, \qquad \kappa(t) = \frac{\left|\frac{d\vec T}{ds}(t)\right|}{|\vec r\,'(t)|} \quad \left(= \frac{|\vec r\,'(t)\times \vec r\,''(t)|}{|\vec r\,'(t)|^3}\right) \text{ for curves in 3-space}$$

Know: $|\vec N(t)| = |\vec T(t)| = 1$ & $\vec N(t) \perp \vec T(t)$.

These are enough to describe motions in $\mathbb{R}^2$, but not in $\mathbb{R}^3$ since we have ~~room~~ to change a course & twist.

Q: How quickly does a curve move out of the (osculating) plane determined by $\vec T(t)$ & $\vec N(t)$?

§12.9 The Binormal Vector and the Torsion (in $\mathbb{R}^3$)



Def: $\vec B(t) = \vec T(t) \times \vec N(t)$ is the unit binormal vector

($|\vec B(t)| = |\vec T(t)|\,|\vec N(t)| \sin\frac{\pi}{2} = 1$ for all t)

TNB-frame: right-handed coordinate system of three unit pairwise perpendicular vectors.

Def: The twisting out of the osculating plane equals $\frac{d\vec B}{ds}$ (rate of change of $\vec B$ with respect to arc length parameter)

Formula:

$$\frac{d\vec B_1}{ds}(s) = \frac{d}{ds}\left(\vec T_1(s) \times \vec N_1(s)\right) \underset{\text{cross-product rule}}{=} \underbrace{\frac{d\vec T_1}{ds}(s) \times \vec N_1(s)}_{=0 \ \left(\vec N_1 = \frac{1}{\kappa(s)}\frac{d\vec T_1}{ds}(s)\right)} + \vec T_1(s) \times \frac{d\vec N_1}{ds}(t)$$

so $$\boxed{\frac{d\vec B_1}{ds}(s) = \vec T_1(s) \times \frac{d\vec N_1}{ds}}$$ (assuming arc length param)

Note: $\vec T(t(s)) = \vec r_1(s)$ & use chain rule: $\frac{d\vec B}{dt} = \frac{d\vec B_1}{ds}(s(t)) \cdot \frac{ds}{dt}(t) = |\vec r\,'(t)| \frac{d\vec B_1}{ds}(s(t))$

so $$\boxed{\frac{d\vec B_1}{ds}(s) = \frac{1}{|\vec r\,'(t)|}\frac{d\vec B}{dt}(s(t))}$$ and $$\boxed{\frac{d\vec B_1}{ds}(s(t)) = \left(\vec T(t) \times \frac{d\vec N}{dt}(t)\right)\frac{1}{|\vec r\,'(t)|}}$$

Properties: 1) $\frac{d\vec B}{ds} \perp \vec T$ & $\frac{d\vec B}{ds} \perp \vec B$ (because $|\vec B(t)| \equiv 1$), so $\frac{d\vec B}{ds} = (-\tau)\vec N$ and $\tau :=$ torsion (scalar!)

We compute $\tau(t) = -\frac{d\vec{B}}{ds} \cdot \vec{N}(t)$ & $|\tau(t)|$ = rate at which the curve twists out of the TN-plane

Examples: Recitation IV Handout (02/04/16)

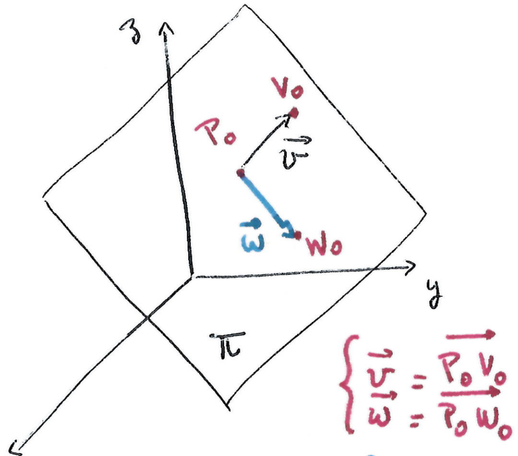
§13. Functions of several variables

§13.1 Planes and surfaces:

§1: Equations of planes in 3-space

We can determine a plane in 2 ways:

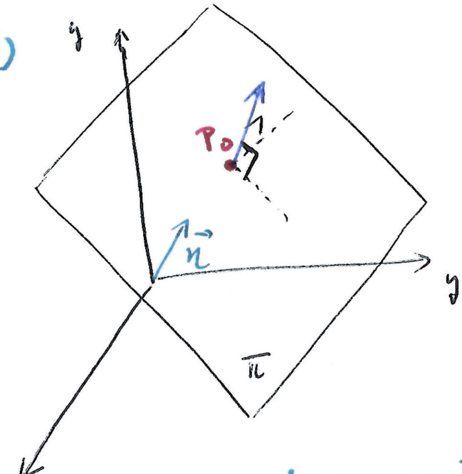
(i)



$$\begin{cases} \vec{v} = \overrightarrow{P_0V_0} \\ \vec{w} = \overrightarrow{P_0W_0} \end{cases}$$

plane defined by a point P_0 & 2 directions $\vec{v}$, $\vec{w}$ (or 3 points P_0, W_0, V_0)

(ii)



plane defined by a point P_0 and a normal vector $\vec{n}$

Notice: $\vec{n} \perp \vec{v}$ & $\vec{n} \perp \vec{w}$.

$$\begin{cases} \vec{n} = \langle a, b, c\rangle \\ Q = \langle x, y, z\rangle. \end{cases} \qquad P_0 = \langle x_0, y_0, z_0\rangle$$

We get a vector & a scalar equation from (ii)

head of Q lies in the plane π if and only if $\overrightarrow{P_0Q} \perp \vec{n}$, so $\boxed{\overrightarrow{P_0Q} \cdot \vec{n} = 0}$ vector eqn

Explicitly: $\langle x-x_0, y-y_0, z-z_0\rangle \cdot \langle a, b, c\rangle = 0$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0} \quad \text{scalar eqn}$$

From (i) $\vec{r}(t,s) = \underbrace{\langle x_0, y_0, z_0\rangle}_{=\overrightarrow{OP_0}} + t\,\vec{v} + s\,\vec{w}$ ..

$$= \overrightarrow{OP_0} + t\,\overrightarrow{P_0V_0} + s\,\overrightarrow{P_0W_0}$$

Example: Find the equation of the plane passing through $(1,0,0)$, $(1,1,1)$ & $(2,1,-1)$. Compute its intercepts (= intersection with the 3 coordinate planes)

- $\vec{r}(t,s) = \overrightarrow{OP_0} + t\,\overrightarrow{P_0V_0} + s\,\overrightarrow{P_0Q} = \langle 1,0,0\rangle + t\langle 0,1,1\rangle + s\langle 1,1,-1\rangle$
- normal $= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = (-2)\vec{i} - (-1)\,\vec{j} + (-1)\,\vec{k} = \langle -2, 1, -1\rangle$ ~ $\langle 2,-1,1\rangle$

So $\vec{Q} = \langle x, y, z \rangle$: $2(x-1) + (-1)(y-0) + 1(z-0) = 0$

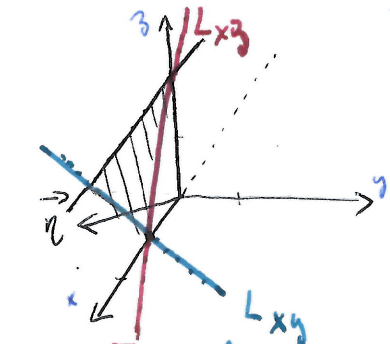
Check: 3 pts satisfy this equation! ✓ $\boxed{2x - y + z = 2}$

The Intercepts will be 3 lines:

xy-plane $\cap\, \pi = \begin{cases} z=0 \\ 2x-y+z=2 \end{cases}$ so line $L_{xy} = \begin{cases} 2x-y=2 \\ z=0 \end{cases}$, so $\boxed{y = 2x-2}$

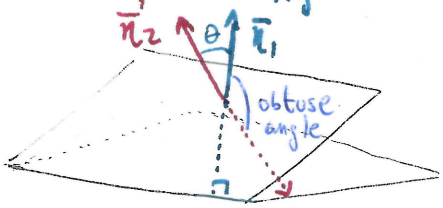
yz-plane $\cap\, \pi = \begin{cases} x=0 \\ 2x-y+z=2 \end{cases}$ so line $L_{yz} = \begin{cases} \boxed{z = 2+y} \\ x=0 \end{cases}$

xz-plane $\cap\, \pi = \begin{cases} y=0 \\ 2x-y+z=2 \end{cases}$ so $L_{xz} \begin{cases} y=0 \\ \boxed{z = 2-2x} \end{cases}$

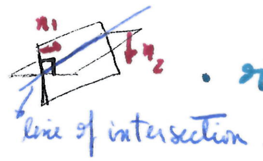


§2. Parallel and Orthogonal Planes

Angle between 2 planes = acute angle between their normal vectors. $(0 \le \theta \le \pi/2)$



In particular: · parallel planes = normal vectors are proportional (parallel)



· orthogonal planes = normals are perpendicular

Note: If $\vec{n}$ is the normal vector to a plane, so is $-\vec{n}$.

Example: Find the parallel plane to $3x - 2y + 5z = 4$ passing through $(1,-1,1)$

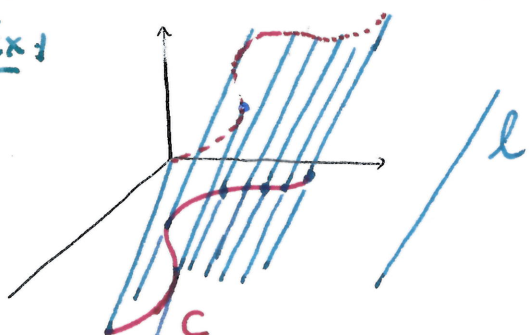
Soln: $0 = \langle 3, -2, 5 \rangle \cdot \langle x-1, y+1, z-1 \rangle = 3(x-1) - 2(y+1) + 5(z-1)$.

So, plane is $\boxed{3x - 2y + 5z = 10}$.

§3 Cylinders and Traces:

Def: Given a curve C in a plane π, and a line ℓ not in this plane, a cylinder is the surface of all lines parallel to ℓ the pass through C.

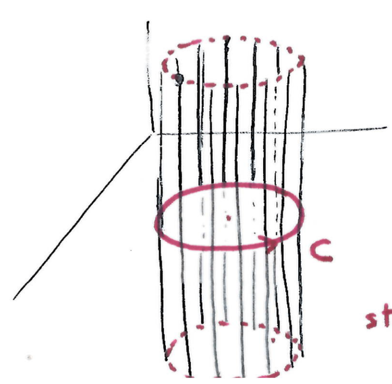
Ex 1



π = xy-plane

Surface is "ruled" by the lines ℓ along C.

Ex 2



ℓ = z-axis

C circle.

standard cylinder (tube)