§3. Cylinders and Traces:

**Def:** given a curve C in a plane, and a line l not in this plane, a **cylinder** is the surface of all lines parallel to l that pass through C.

**Examples:** Assume the plane is the xy-coordinate plane.

1. ![Diagram of a cylinder](image)
   - Surface is "ruled" by the line l along C.

2. ![Diagram of a standard cylinder](image)
   - **Def:** A Trace of a surface is the set of points at which the surface intersects a plane that is parallel to a coordinate plane. We call them xy-traces, yz-traces, and xz-traces, accordingly.

3. ![Examples of traces](image)
   - **Why?** Use any trace to sketch the surface. (Ex: above: elliptic paraboloid)

- We will use this to sketch quadratic surfaces.
- **Special traces: Intercepts** = traces at the standard coordinate planes (x=0, y=0, z=0, etc.)
5.2 Quadratic surfaces

A quadratic surface in 3-space is given by a general equation of degree 2:

\[ A x^2 + B y^2 + C z^2 + D x y + E x z + F y z + G x + H y + I z + J = 0 \]

where \( A, B, \ldots, J \) are fixed constants and not all \( A, B, \ldots, F \) are zero.

To draw them: use 3 intercepts & at least 2 extra traces of each kind.

By changing coordinates, we get 6 standard examples (Table 13.1 [page 901]).

We do examples of all these surfaces (HW: One of each type). All traces will be conic sections (ellipses, parabolas or hyperbolas).

1. **ELLIPSOID**

   \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad a, b, c > 0. \]

   **Example:** \( \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1 \)

   The role of all 3 variables is symmetric.

   **Xy-Traces:** \( z = 0 \)

   \[ \frac{x^2}{9} + \frac{y^2}{16} = 1 \]

   **Xz-Traces:** \( y = 0 \)

   \[ \frac{x^2}{9} + \frac{z^2}{25} = 1 \]

   **Yz-Traces:** \( x = 0 \)

   \[ \frac{y^2}{16} + \frac{z^2}{25} = 1 \]

   The xy-traces are symmetric with respect to \( z = 0 \).

   \( z = \pm 1 \)

   \[ \frac{x^2}{9} + \frac{y^2}{16} + \frac{1}{25} = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = \frac{1 - \frac{1}{25}}{25} \]

   \( z = \pm 5 \) we get a point \( (0, 0, \pm 5) \)

   \[ 125 = 5 \]

   **Conclusion:**

2. **ELLIPITIC PARABOLOID**

   \[ z = \frac{x^2 + y^2}{a^2 + b^2} \quad a, b > 0 \]

   **Notice** \( z \geq 0 \) and \( z = 0 \) if and only if \( x = y = 0 \).

   For every \( z > 0 \), the xy-traces are ellipses, that grow away from \( (0, 0) \).
1. $x^2 + y^2 = z^2$ paraboloid, "slope $= \frac{1}{k}$".

2. In general, $x = \pm x_0$, $z = \frac{x_0^2}{a^2} + \frac{y^2}{b^2}$ paraboloid, shifted up by $\frac{x_0^2}{a^2}$ units.

3. Similar for $y$ and $z$-traces.

4. Lowest paraboloid when $y = 0$ and shifted up by $\frac{y_0^2}{a^2}$.

5. Conclusion: $x^2 + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

6. Hyperboloid of One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

7. $x^2 + \frac{y^2}{16} - \frac{z^2}{25} = 1$.

8. Symmetric behavior on $x$ and $y$, but different for $z$.

9. Notice that it is unbounded in the $z$-direction, and symmetric.

3. Hyperboloid of One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

Xz-Trace: $x = 0$:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$y = 0$:

$\frac{y^2}{b^2} = 1$.

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + 1 = 2$ ellipse.

$y = 0$:

$\frac{x^2}{a^2} - \frac{y^2}{c^2} = 1$ hyperbola.

$\{(x = \pm a, y = 0)\}$ vertices.

$\{(x = \pm a, y = 0)\}$ asymptotes:

$y = \pm \frac{c}{a}x$.
**Hyperboloid of Two Sheets**

Equation: 
\[ -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad a, b, c > 0 \]

Notice: 
- Traces with \( z = \pm c \) and \( |z| < c \) are empty.
- \( xz\)-trace with \( z = \pm c \): \( (a, 0, \pm c) \) is an ellipse.

\( xz\)-trace: 
- When \( |z| > c \): \[ -\frac{x^2}{a^2} + \frac{y^2}{b^2} > 0 \] is a hyperbola with asymptotes \( z = \pm \frac{3}{5} x \)

Conclusion:

\( xz\) and \( yz\)-traces show a similar behavior.

Conclusion:

- \( xz\)-trace: \[ -\frac{x^2}{a^2} + \frac{y^2}{b^2} > 0 \] is a hyperbola with asymptotes \( z = \pm \frac{3}{5} x \)
(5) **ELLIPITIC CONE**

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \]

- \(x, y\)-trace are ellipses (\(c=0\) gives the pt \((0,0)\)).
- \(x, y\)-trace, \(y, z\)-trace have the same behavior.

- \(x, y\)-trace: \(y = 0\)
  \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = \left( \frac{x}{a} \right) \left( \frac{y}{b} \right) = 0 \]
  \(\text{uninf. 2 lines.}\)

- \(y > 0\)
  \[ \frac{x^2}{16} = \frac{z^2}{c^2} - \frac{x^2}{a^2} \]
  \(\text{hyperbolic with asymptotes } x = \pm \frac{3}{4} \)
  \(\text{as } |z| \text{ grows, the hyperbola moves away from the asymptotes.}\)

**Conclusion:**

![Elliptic Cone Diagram]

(6) **HYPERBOLIC PARABOLOID:**

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = z. \]  (compare with (5))

- \(x, y\)-trace:
  \[ z = \frac{x^2}{a^2} - \frac{y^2}{b^2} = z_0 \]
  \(z_0 = 0\)
  \(\text{get a unin. 2 lines, } x = \frac{3}{4} y.\)
  \(z_0 > 0\)
  \(\text{hyperbola moving away from 2 lines.}\)
  \(z_0 < 0\)
  \(\text{moving away from 2 lines.}\)

- \(x, y\)-trace:
  \(x = x_0\)
  \(\text{get a parabola with slope } \frac{1}{4} < 0.\)

- \(y, z\)-trace:
  \(y = y_0\)
  \(\text{with slope } \frac{1}{4} > 0\)
  \[ z = \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{y^2}{16} \]

**Conclusion:**

![Hyperbolic Paraboloid Diagram]