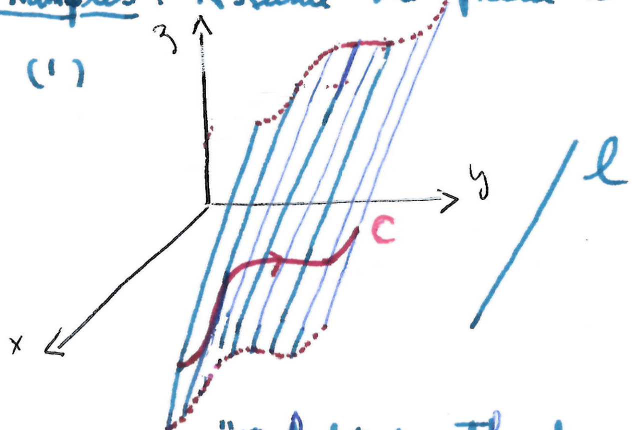
§3. Cylinders and Traces:

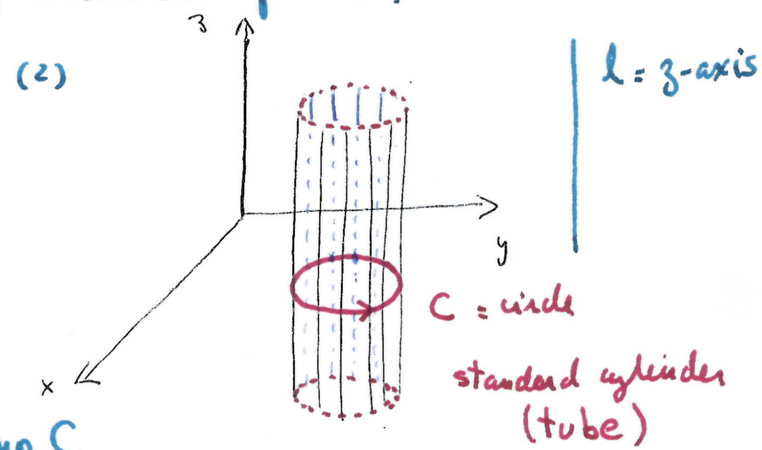
Def: Given a curve C in a plane, and a line ℓ not in this plane, a <u>cylinder</u> is the surface of all lines parallel to ℓ that pass through C.

Examples: Assume the plane is the xy-coordinate plane.

(1)



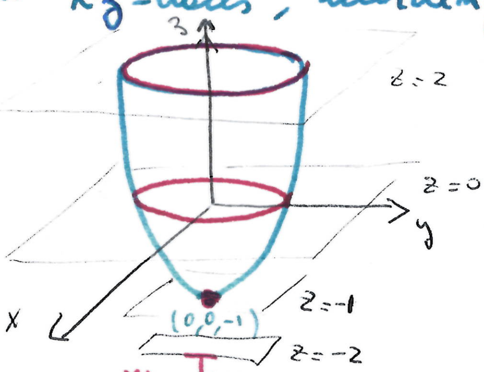
(2)



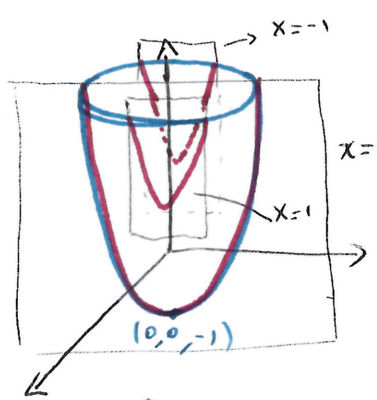
Surface is "ruled" by the line ℓ along C.

Def A <u>Trace</u> of a surface is the set of points at which the surface intersects a plane that is parallel to a coordinate plane. We call them xy-traces, yz-traces and xz-traces, accordingly.

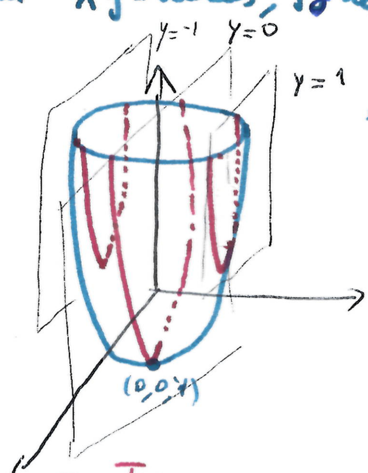
Ex:



xy-traces

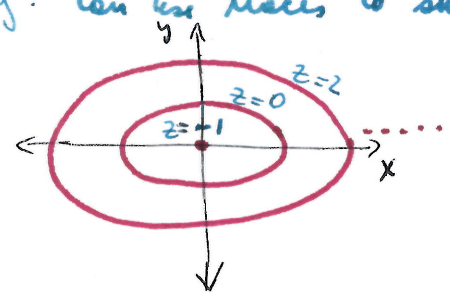


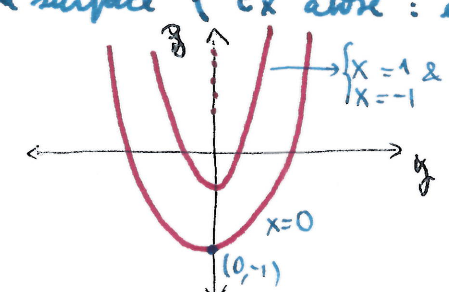
yz-traces

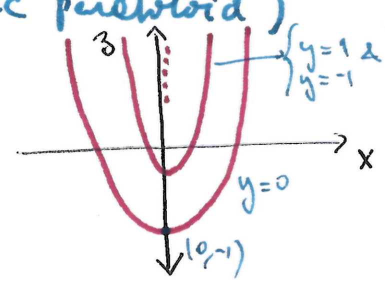


xz-traces

Why? Can use traces to sketch the surface (Ex above: elliptic paraboloid)







We will use this to sketch <u>quadratic surfaces</u>

<u>Special traces</u>: <u>intercepts</u> = traces at the standard coordinate planes (z=0, x=0 & y=0, resp.)

§2 Quadratic surfaces

A quadratic surface in 3-space is given by a general equation of degree 2:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where A, B, ..., J are fixed constants and not all A, B, ..., F are zero.

- To draw them: use 3 intercepts & at least 2 extra traces of each kind.

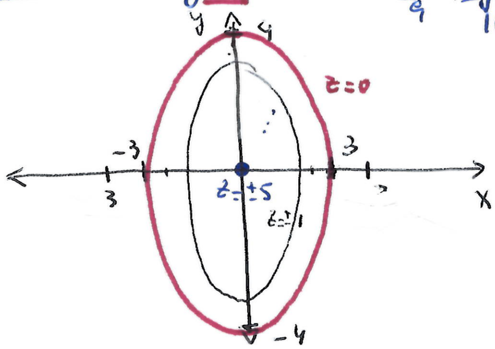
By changing coordinates, we get 6 standard examples (Table 13.1 [page 901])

- We do examples of all these surfaces (HW: One of each type). All traces will be conic sections (ellipses, parabolas or hyperbolas)

① ELLIPSOID. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $a, b, c > 0$.

Example: $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1$ The role of all 3 variables is symmetric.

xy-traces. $z=0$ $\frac{x^2}{9} + \frac{y^2}{16} = 1$ Ellipse Draw intercepts: $(\pm 3, 0)$ $[(\pm a, 0)]$
$(0, \pm 4)$ $[(0, \pm b)]$



xy-traces are symmetric with respect to $z=0$.

$z = \pm 1$ $\frac{x^2}{9} + \frac{y^2}{16} + \frac{1}{25} = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1 - \frac{1}{25}$

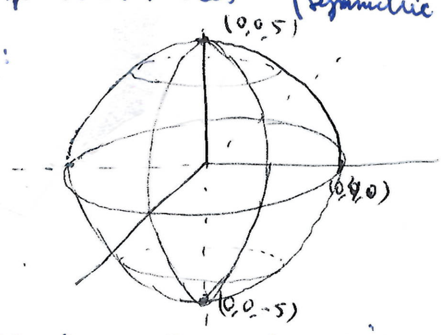
$\Rightarrow$ smaller ellipse.

Highest value for $|z|$ to have non-empty trace: $\frac{z^2}{25} = 1$

$z = \pm 5$ we get a point $= (0, 0, \pm 5)$ $\boxed{|z| = 5}$

Similar for other traces (symmetric in x, y, z).

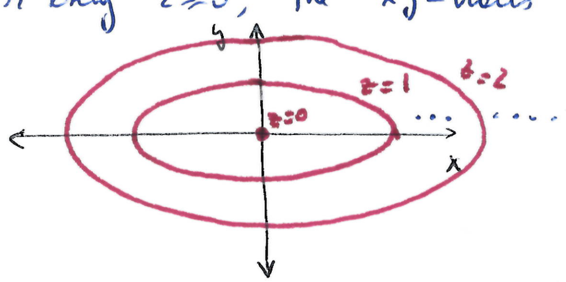
Conclusion:



② ELLIPTIC PARABOLOID $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ $a, b > 0$ Eg: $z = \frac{x^2}{4} + \frac{y^2}{4}$

Notice $z \geq 0$ and $z = 0$ if and only if $x = y = 0$.

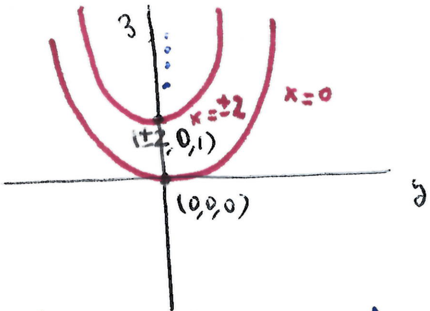
- For every $z \geq 0$, the xy-traces are concentric ellipses, that grow away from $(0,0)$.



xz-trace & yz-trace will have same behavior:

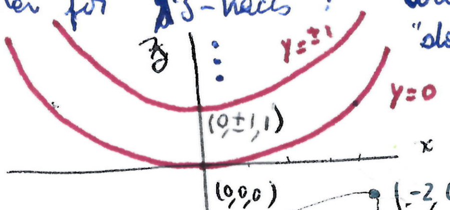
$\underline{x=0}$: $z = \frac{y^2}{1}$ parabola "slope $= \frac{1}{b^2}$".

in general $x = \pm x_0$ $z = \underbrace{\frac{x_0^2}{4}}_{\text{constant}} + \frac{y^2}{1}$ parabola shifted up by $\frac{x_0^2}{4}$-units

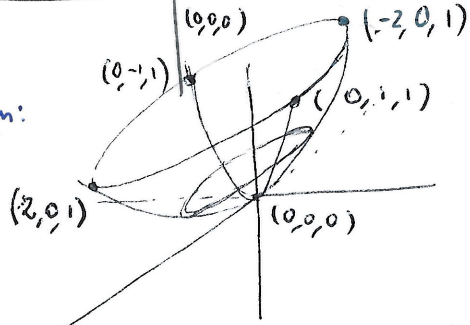


Similar for yz-traces: lowest parabola when $y=0$. Rest, shifted up by $\frac{y_0^2}{1}$.

"slope" $= \frac{1}{a^2} = \frac{1}{4}$.



Conclusion:



③ HYPERBOLOID OF ONE SHEET:

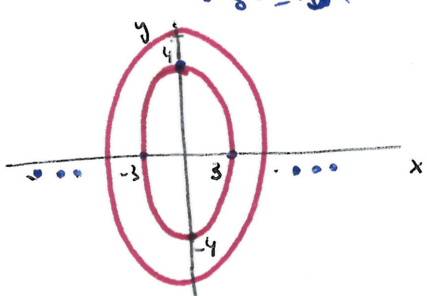
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad a, b, c > 0.$$

Eg: $\frac{x^2}{9} + \frac{y^2}{16} - \frac{z^2}{25} = 1$.

- Symmetric behavior on x & y, but different for z. (also: symmetry with respect to origin for each trace) and symmetric.
- Notice that it is unbounded in the z-direction

xy-traces: $\cdot\, z=0$: $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

$\cdot\, z = \pm 5$: $\frac{x^2}{9} + \frac{y^2}{16} = 1+1 = 2$ so ellipses are concentric and grow away from $(z=0)$ trace..

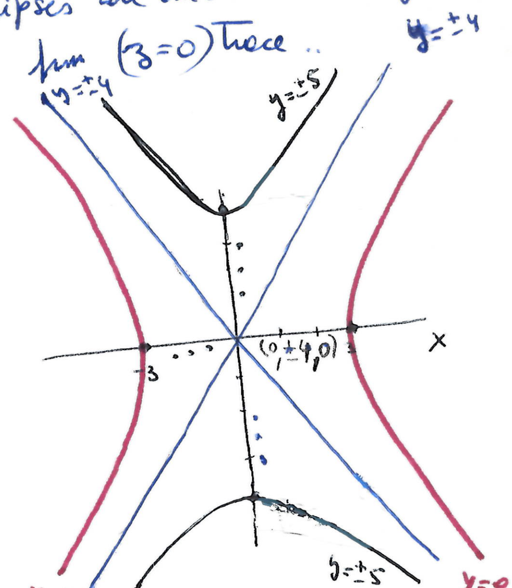


xz-traces $\cdot\, y=0$ $\frac{x^2}{9} - \frac{z^2}{25} = 1$ hyperbola

$\begin{cases} \cdot\, (x = \pm 3,\ z = 0) & \text{2 vertices} \\ \cdot\, \text{asymptotes: } z = \pm \frac{5}{3} x \end{cases}$

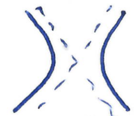
$\cdot\, y = \pm 4$. $\frac{x^2}{9} - \frac{z^2}{25} = 0 \to \left(\frac{x}{3} - \frac{z}{5}\right)\left(\frac{x}{3} + \frac{z}{5}\right) = 0$

union of 2 lines (asymptotes from before!)



Fix $y = \pm y_0$: $(y_0 \geq 0)$ $\quad \frac{x^2}{9} - \frac{z^2}{25} = \left(1 - \frac{y_0^2}{16}\right)$ → check sign!

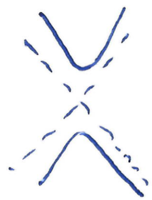
- $0 \leq y_0 < 4$ $\quad$ sign > 0. hyperbolas



- $y_0 = 4$ $\quad$ sign $= 0$

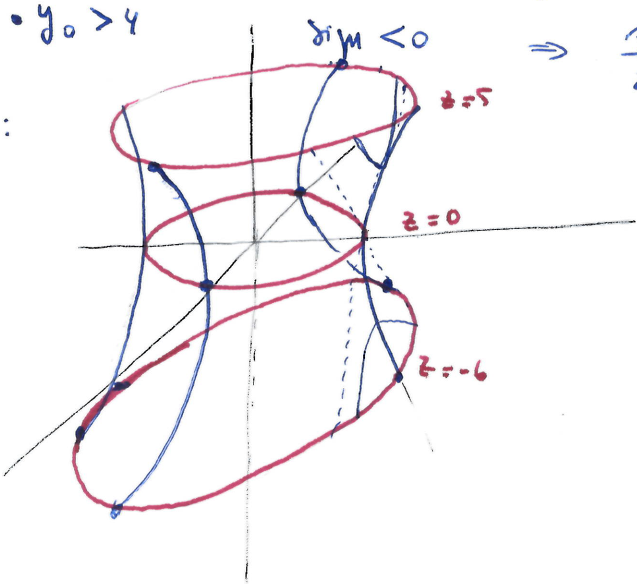
union of 2 lines.

- $y_0 > 4$ $\quad$ sign < 0 $\Rightarrow \frac{z^2}{25} - \frac{x^2}{9} = \underbrace{\frac{y_0^2}{16} - 1}_{>0}$



hyperbolas

Conclusion:



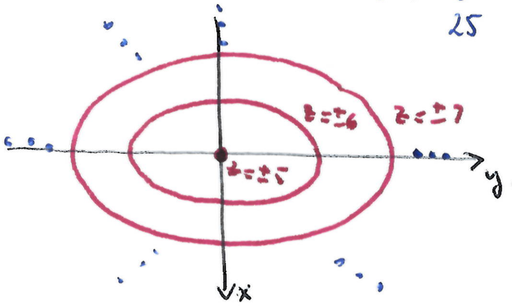
④ HYPERBOLOID OF TWO SHEETS

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad a, b, c > 0$$

Eg: $-\frac{x^2}{9} - \frac{y^2}{16} + \frac{z^2}{25} = 1.$

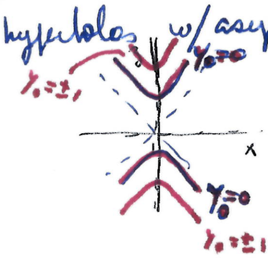
Notice:
- Traces with $z = \pm z_0$ and $|z_0| < 5$ are empty!
- xy-trace with $z = \pm 5$: $(0,0,\pm 5)$ 1 pt.

xy-traces
- When $|z_0| > 5$: $0 < \frac{z_0^2}{25} - 1 = \frac{x^2}{9} + \frac{y^2}{16}$ → so get concentric ellipses that grow away from $(0,0)$.

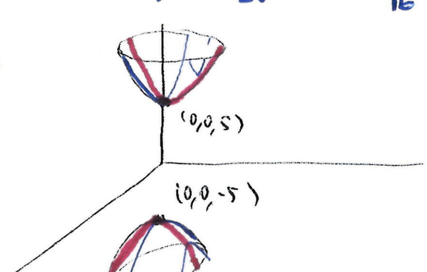


xz & yz-traces have a similar behavior

xz-traces: $\frac{-x^2}{9} + \frac{z^2}{25} = 1 + \frac{y_0^2}{16} > 0$ → so traces are hyperbolas w/ asymptotes $z = \pm \frac{3}{5} x$



Conclusion:



(5) ELLIPTIC CONE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

Eg: $\frac{x^2}{9} + \frac{y^2}{16} = \frac{z^2}{25}$

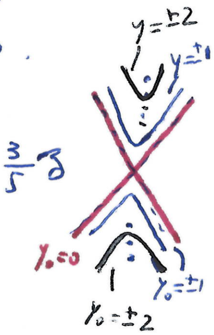
All xy-traces are ellipses ($z=0$ gives the pt $(0,0,0)$).

- xz-traces + yz-traces have the same behavior

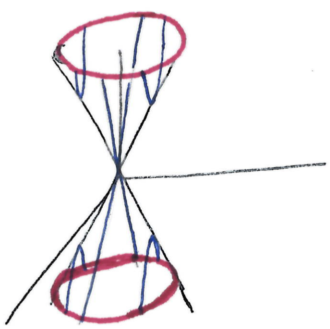
xz-traces: $y_0 = 0$. $\frac{x^2}{9} - \frac{z^2}{25} = \left(\frac{x}{3} - \frac{z}{5}\right)\left(\frac{x}{3} + \frac{z}{5}\right) = 0$ union of 2 lines.

$y_0 > 0$ $\frac{y^2}{16} = \frac{z^2}{25} - \frac{x^2}{9}$: hyperbola w/ asymptotes $x = \pm\frac{3}{5}z$

As $|y_0|$ grows, the hyperbola moves away from the asymptotes



Conclusion:



(6) HYPERBOLIC PARABOLOID:

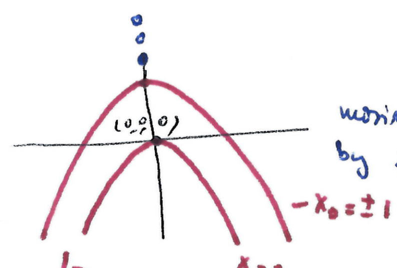
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z.$$ (compare with (2))

Eg: $z = \frac{x^2}{9} - \frac{y^2}{16}$.

Notice: Symmetry between x & y is broken by the different signs.

xy-traces: $z = z_0$:
- $z_0 = 0$ get a union of 2 lines $x = \pm\frac{3}{4}y$.
- $z_0 > 0$ " " hyperbola ... w/ asymptotes $z_0=0$ trace. (moving away from 2 lines)
- $z_0 < 0$ " " (moving away from 2 lines)

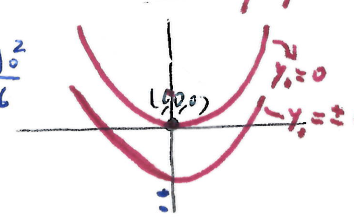
xz-traces: $x = x_0$: get a parabola with slope $-\frac{1}{16} < 0$ $z = \left(\frac{x_0^2}{9}\right) - \frac{y^2}{16}$



moving up by $\frac{x_0^2}{9}$ units

yz-traces: $y = y_0$: with "slope" $1/9 > 0$ $z = \frac{x_0^2}{9} - \frac{y_0^2}{16}$

moving down $\frac{y_0^2}{16}$ units



Conclusion: