Lecture XIII (2/10/16): §13.2 Graphs & level curves, §13.3 Limits and continuity

§13.2

§1 Functions of 2 variables

- Explicit form: $(x,y) \longmapsto f(x,y) = z$ ((x,y) in some set D = domain of f)
- Implicit form: $F(x,y,z) = 0$ [eg quadratic surfaces]

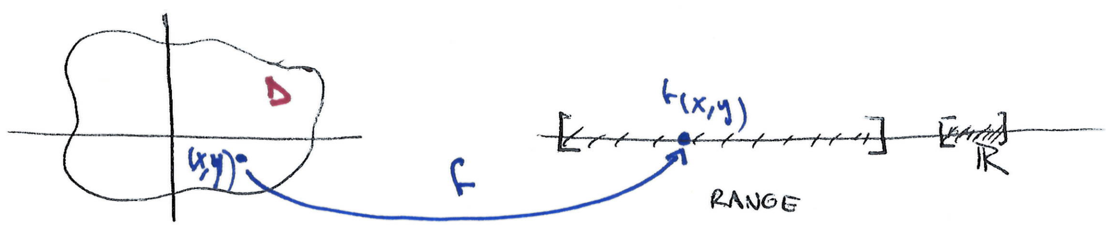
Eg planes: Explicit $(s,t) \longmapsto -(\frac{1}{2}s + \frac{3}{4}t)$

Implicit $2x + 3y + 4z = 0$.

Def: • A function $z = f(x,y)$ assigns to each point (x,y) in a set D in $\mathbb{R}^2$ a unique number z in a subset of $\mathbb{R}^n$.

[For functions of n variables: $x_{n+1} = f(x_1, x_2, \dots, x_n)$
- Domain = set in $\mathbb{R}^n$
- Range: possible values of x_{n+1} in $\mathbb{R}$]

- The domain of f is the set D
- The range (codomain, or image) is the set of all z in $\mathbb{R}$ that can be written as $f(x_0,y_0)$ for some (x_0,y_0) in D.



Eg: Find the domain of $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ & the range.

Soln: $u = \sqrt{x^2+y^2}$ cannot be 0 so $x^2+y^2 \neq 0$ is the domain.

$D = \{(x,y) \text{ in } \mathbb{R}^2 \mid (x,y) \neq (0,0)\} = \boxed{\mathbb{R}^2 \setminus \{(0,0)\}}$

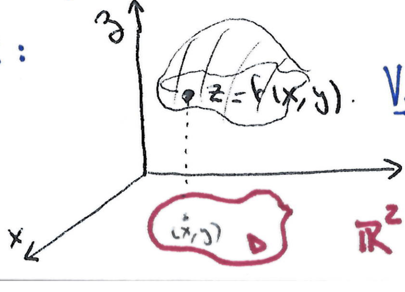
Range = The range of $g(u) = \frac{1}{u}$ is $\mathbb{R} \setminus \{0\}$, & the range of

$u(x,y) = \sqrt{x^2+y^2}$ when $(x,y) \neq (0,0)$ is $\mathbb{R}_+ \setminus \{0\}$ = domain of g.

Answer the question by composing maps:

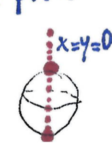
$\boxed{\text{Range} = \mathbb{R}_+ \setminus \{0\} = \mathbb{R}_{>0}}$

§2 Graphs:



V. Line Test: Every vertical line meets the surface $z = f(x,y)$ at EXACTLY ONE POINT (Otherwise, a point (x_0,y_0) would have 2 images under f)

- Ellipsoid: fails the test (eg: $x^2+y^2+z^2=1$).

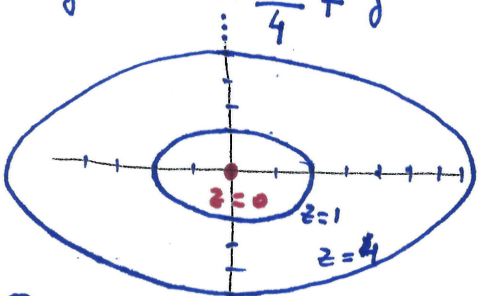


• Elliptic paraboloid: passes the test (Eg. $z = \frac{x^2}{4} + y^2$)

§3 Level curves: Eg Topographic maps.

Def: Pick $z = f(x,y)$ a function & draw its graph in $\mathbb{R}^2$. A level curve at $z = z_0$ is the xy-Trace of the surface at $z = z_0$.

Eg: $z = \frac{x^2}{4} + y^2$ Elliptic paraboloid from Lecture XII.



... 3 level curves. Concentric ellipses centered at (0,0) that grow away as $z_0 \to +\infty$.

• Q: Can two different level curves intersect?

A: NO Otherwise $z_0 = f(x_0, y_0) = z_1$ where (x_0, y_0) is the intersection pt of the 2 level curves, but $z_0 \neq z_1$

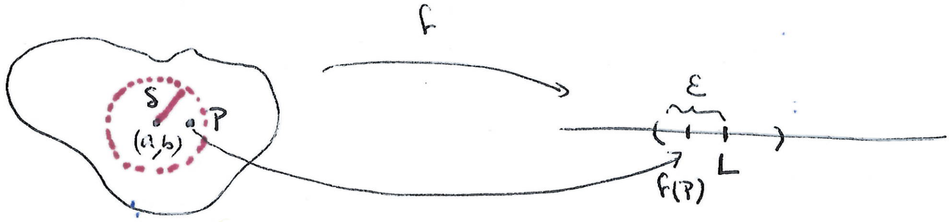
• Level curves can change shapes with z_0 (see Fig 13.34)

§13.3

§1 Limit of a function of two variables:

Def: A function $f: D \to \mathbb{R}$ has a limit L as $P = (x,y)$ approaches a fixed point $P_0 = (a,b)$ if $|f(x,y) - L|$ can be made arbitrarily small for all P in D that are sufficiently close to P_0. If the limit exists, we write $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

Q: What does "close to" mean?

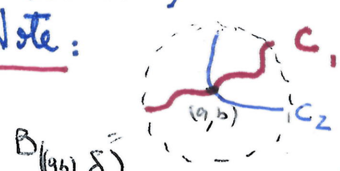


dist $(L, f(P)) = |f(P) - L|$ small? Given $\varepsilon > 0$, want to find $\delta > 0$ ($\delta(\varepsilon)$, small) for all P in D if $|\overrightarrow{PP_0}| < \delta$, then $|f(P) - L| < \varepsilon$

$$|\overrightarrow{PP_0}| = \sqrt{(x-a)^2 + (y-b)^2}$$

(CURVE TEST)

Note:



If a limit L exists, then $L = \lim_{\substack{(x,y)\to P_0 \\ (x,y) \text{ in } C_1}} f(x,y) = \lim_{\substack{(x,y)\to P_0 \\ (x,y) \text{ in } C_2}} f(x,y)$

So the limit is the same if we approach (a,b) along any curve C in D passing through (a,b).

This remark will be useful to show a given limit does not exist.

Example $f(x,y) = \dfrac{x}{y}$ $\quad D = \{(x,y) \mid y \neq 0\} = \mathbb{R}^2 \setminus \{x\text{-axis}\}$.

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$ $\quad$ Try with curve $C_1: \vec{r}(x) = \langle x, x\rangle$, $C_2: \vec{r}(y) = \langle 0, y\rangle$

- $\lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y) \text{ in } C_1}} f(x,y) = \lim\limits_{x\to 0} f(x,x) = \lim\limits_{x\to 0} \dfrac{x}{x} = \lim\limits_{x\to 0} 1 = 1$
- $\lim\limits_{\substack{(x,y)\to(0,0)\\ (x,y) \text{ in } C_2}} f(x,y) = \lim\limits_{y\to 0} f(0,y) = \lim\limits_{y\to 0} 0 = 0$

$0 \neq 1$ so the original limit does NOT exist.

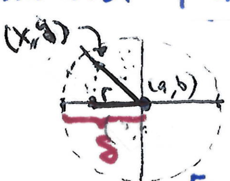
Theorem 1: Some useful limits

(1) $\lim\limits_{(x,y)\to(a,b)} c = c$ $\quad$ (limit of constant function)

(2) $\lim\limits_{(x,y)\to(a,b)} x = a$ & $\lim\limits_{(x,y)\to(a,b)} y = b$. $\quad$ (limit of linear functions)

Pf/ (1) Pick $\varepsilon > 0$: $f(P) = c$ for all P so $|f(P) - c| = 0 < \varepsilon$ for any P, so we can take $\delta = \varepsilon$ (any $\delta > 0$ would do). ✓

(2) Pick $\varepsilon > 0$. $\quad f(P) = x \quad |f(P) - a| = |x - a| < \varepsilon$ is our goal whenever $|\langle x-a, y-b\rangle| < \delta$. We must find $\delta > 0$ ensuring our goal.



want $r < \varepsilon$ if $r < \delta$. Pick $0 < \delta \leq \varepsilon$ ✓

Theorem 2: [Algebra of limits] Pick L, M in $\mathbb{R}$ where $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$. Assume c in $\mathbb{R}$ is a constant, m, n in $\mathbb{Z}$ (integers) with no common factors & $n \neq 0$.

(1) SUM & (2) DIFFERENCE $\lim\limits_{(x,y)\to(a,b)} (f \pm g)(x,y) = L \pm M$

(3) CONSTANT MULTIPLE $\lim\limits_{(x,y)\to(a,b)} c f(x,y) = cL$.

(4) PRODUCT: $\lim\limits_{(x,y)\to(a,b)} f(x,y)\, g(x,y) = LM$.

(5) QUOTIENT: If $M \neq 0$, $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$

(6) POWER: $\lim\limits_{(x,y)\to(a,b)} (f(x,y))^m = L^m$, $\lim\limits_{(x,y)\to(a,b)} (f(x,y))^{m/n} = L^{m/n}$ ($L > 0$ if n even)