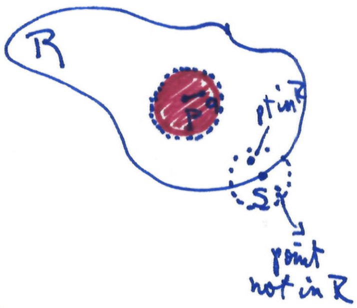
Consequence (last time) Polynomials have limits at every point (a,b). Eg $P(x,y) = x^4 - 3xy + y^5 + 10$.

§2 Limits at Boundary Points:

Def: Let R be a region in $\mathbb{R}^2$. A point P in R is an interior point of R if we can fix $\delta > 0$ where $|PQ| < \delta$ ensures Q is in R.



A point S in R is a boundary point of R if every disc $B(S,\varepsilon)$ (where $\varepsilon > 0$ is arbitrary) centered a S contains at least one point in R and at least a point outside R. $[B(S,\varepsilon) = \{P : |\vec{PS}| < \varepsilon\}]$

Def: • A region is OPEN if all its points are interior points. (Eg $B(0,1)$)

• A region is CLOSED if it contains all its boundary points (Eg ▨ $[0,1]$ square.)

• We can take limits at interior pts and ALSO at boundary pts.

§3 Continuity of Functions of 2 variables

Def: The function $f : D \to \mathbb{R}$ is continuous at (a,b) provided:

(1) f is defined at (a,b) (that is (a,b) lies in D)

(2) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists

(3) This limit equals $f(a,b)$.

Example:
$$f(x,y) = \begin{cases} \dfrac{8xy^2}{x^2+y^4} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$
Where is f continuous?

Soln: • Continuity outside $(0,0)$ holds because x^2+y^4 will never be 0 in this region, so the rational function is continuous in its domain $= \mathbb{R}^2 \setminus \{(0,0)\}$.

Continuity at $(0,0)$? We check the limits along paths:

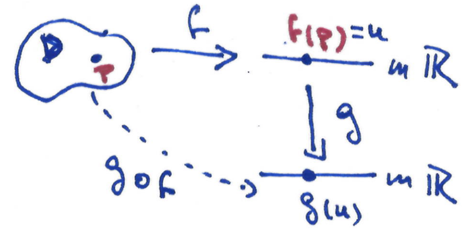
1) LINEAR PATHS through $(0,0)$: $y = mx$ for some m (slope)
$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{8x(mx)^2}{x^2+m^4x^4} = \lim_{x\to 0} \frac{8m^2x^3}{x^2(1+m^2x^2)} \underset{x\neq 0}{=} \lim_{x\to 0} \frac{8m^2x}{1+m^2x^2} = 0.$$

2) QUADRATIC PATHS through $(0,0)$: $x = my^2$ for some $m \neq 0$
$$\lim_{y\to 0} \frac{8my^2y^2}{m^2y^4+y^4} = \lim_{y\to 0} \frac{8my^4}{y^4(1+m^2)} = \lim_{y\to 0} \frac{8m}{1+m^2} = \boxed{\frac{8m}{1+m^2}} \neq 0$$

So by the path limit test, we conclude the limit does not exist!

- Why did we choose these paths? To cancel the contributions of the numerator & denominator!

Q: What about composing continuous functions?



THEOREM: If $f: D \longrightarrow \mathbb{R}$ is continuous at (a,b) and $g: \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $f(a,b)$, then $g\circ f_{(x,y)} = g(f_{(x,y)})$ is continuous at (a,b).

Applications: g = exponential, trig functions, inverse trig functions, $\sqrt{u}$ if $f(a,b)>0$, $\ln u$, $\frac{1}{u}$ if $f(a,b)\neq 0$ ...

Eg $\tan^{-1}\left(\frac{1}{x+y}\right)$ is composition of cont. functions except when $y=-x$.

§13.4. Partial derivatives:

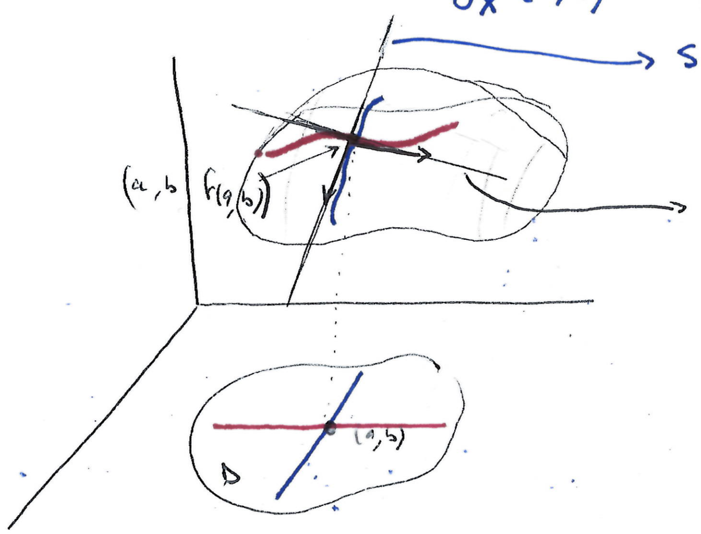
§1 Derivatives with Two variables:

Let $f: D \longrightarrow \mathbb{R}^2$ be a function of 2 variables. Let (a,b) be an interior point in D.

Def: $$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \quad \text{(partial deriv w.r.t. } x)$$

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h} \quad (\text{———————— } y)$$

Alternate notation: $f_x(a,b) = \frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial x}\Big|_{(a,b)}$, $f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = \frac{\partial f}{\partial y}\Big|_{(a,b)}$



SLOPE = $f_x(a,b)$
= rate of change of f w.r.t. x.
= ———————— $f(x,b)$

SLOPE = $f_y(a,b)$
= rate of change of f w.r.t. y.
= ———————— $f(a,y)$
(1 var function!)