Lecture XV (2/15/16) §13.4 (cont.) Partial derivatives

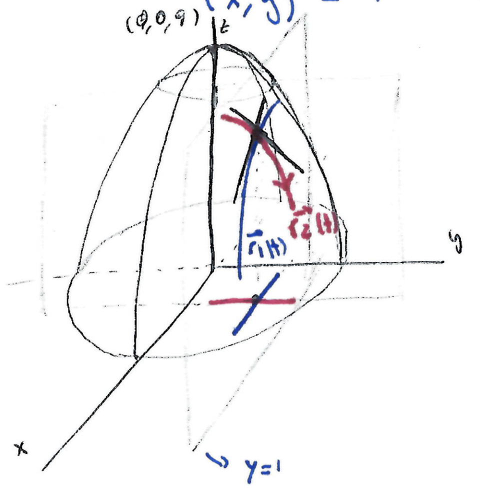
§1 Derivatives with two variables:

Fix $f: D \longrightarrow \mathbb{R}$ be a function of 2 variables and (a,b) an interior point in D.

Def: $f_x(a,b) = \lim_{h\to 0} \dfrac{f(a+h,b)-f(a,b)}{h}$ = rate of change of $g(x) := f(x,b)$ at $x=a$

$f_y(a,b) = \lim_{h\to 0} \dfrac{f(a,b+h)-f(a,b)}{h}$ = ———————— $q(y) := f(a,y)$ at $y=b$.

Example: $f(x,y) = 9-x^2-2y^2$ $\quad (a,b)=(1,1)$ $\quad D=\mathbb{R}^2$ $\quad$ Elliptic Paraboloid



$g(x) = f(x,1) = 9-x^2-2 = 7-x^2$. (parabola)

so $g'(x) = -2x$, $g'(1) = -2 = f_x(1,1)$

$q(y) = f(1,y) = 8-2y^2$ (parabola)

so $q'(y) = -4y$, $q'(1) = -4 = f_y(1,1)$

Note: • Substituting $x=a$ on f is the SAME as taking the intersection of $z=f(x,y)$ with the plane $x=a$ $\equiv$ yz-trace of the graph of f at $x=a$.

• Similarly, setting $y=b$ is the SAME as taking the intersection of $z=f(x,y)$ with the plane $y=b$ $\equiv$ xz-trace of the graph of f at $y=b$

Such intersections give parametric curves

Examples: $z = 9-x^2-2y^2$ $\qquad f(1,1)=6$

$y=1$

$z = 7-x^2$ is param curve

$\vec{r}_1(t) = \langle t, 1, 7-t^2\rangle$

$\vec{r}_1{}'(t) = \langle 1, 0, -2t\rangle$

$x=1$

$z = 8-2y^2$ is param curve.

$\vec{r}_2(t) = \langle 1, t, 8-2t^2\rangle$

$\vec{r}_2{}'(t) = \langle 0, 1, -4t\rangle$

At the point $(1,1,6)$, we get 2 tangent vectors to the surface $\langle 1, 0, \underset{f_x(1,1)}{-2}\rangle$ & $\langle 0, 1, \underset{f_y(1,1)}{-4}\rangle$

[These will be the 2 directions of the tangent plane to the graph of f at $(1,1,6)$]

• By construction, f_x, f_y are again functions (we can vary (a,b)), but their domains might be smaller than D = domain of f.

$$f_x(x,y) := \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f}{\partial x}(x,y)$$

$$f_y(x,y) := \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \frac{\partial f}{\partial y}(x,y).$$

Example: ① $f(x,y) = \ln(x^2+y^2) \rightsquigarrow \frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}$

② $f(x,y) = e^{-x^2-y^2}xy \Rightarrow \frac{\partial f}{\partial x} = xy\, e^{-x^2-y^2}(-2x) + e^{-x^2-y^2}y$ (Prod Rule!)

$\frac{\partial f}{\partial y} = xy\, e^{-x^2-y^2}(-2y) + e^{-x^2-y^2}x.$

Remarks: (1) In order to compute f_x, we treat y as a constant, and in order to compute f_y, we treat x is a constant

(2) Similarly, we can define partial derivatives of a function with n variables $f(x_1,\dots,x_n)$ $(n\geq 2)$

Eg ($n=4$) $\frac{\partial f}{\partial x_3} = \lim_{h\to 0} \frac{f(x_1,x_2,x_3+h,x_4)-f(x_1,x_2,x_3,x_4)}{h}$ & we compute it by treating x_1, x_2 & x_4 as constants.

Eg. $f(x_1,x_2,\dots,x_n) = \sqrt{x_1^2+x_2^2+\dots+x_n^2} \Rightarrow \frac{\partial f}{\partial x_i} = \frac{x_i}{\sqrt{x_1^2+\dots+x_n^2}}$ $i=1,\dots,n$.

§2 Higher-order derivatives: "Take partial derivatives in succession".

• $f_{xx} = (f_x)_x = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial^2 x}$

$\quad \| $

$\lim_{h\to 0} \frac{f_x(x+h,y)-f_x(x,y)}{h}$ ((x,y) must be an interior pt of the domain of f_x)

• $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} := \lim_{h\to 0}\frac{f_x(x,y+h)-f_x(x,y)}{h}$

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Warning: Indices are read in different order for each notation!

Example: $f(x,y) = x^3+3x^2y+y^6+9$

• $f_x = 3x^2+6xy \Rightarrow f_{xx} = 6x+6y$, $f_{xy} = 6x$ $f_{xyx}=6$ $f_{xyy}=0$

• $f_y = 3x^2+6y^5 \Rightarrow f_{yx} = 6x$, $f_{yy} = 30y^4$ $f_{yyy}=120y^3$, etc.

§3 Equality of mixed partial derivatives:

Thm [Clairaut's cross derivative Test]

$f_{xy} = f_{yx}$ if BOTH f_{xy} & f_{yx} are continuous.

Proof (sketch):

$$f_{xy}(a,b) = \lim_{s\to 0}\lim_{h\to 0} \frac{\overbrace{\frac{f(a+h,b+s) - f(a,b+s)}{h}}^{f_x(a,b+s)} - \overbrace{\left(\frac{f(a+h,b) - f(a,b)}{h}\right)}^{f_x(a,b)}}{s}$$

$$= \lim_{s\to 0}\lim_{h\to 0} \frac{f(a+h,b+s) - f(a+h,b) - f(a,b+s) + f(a,b)}{h\cdot s}$$

$$f_{yx}(a,b) = \lim_{h\to 0}\lim_{s\to 0} \frac{\overbrace{\frac{f(a+h,b+s) - f(a+h,b)}{s}}^{f_y(a+h,b)} - \overbrace{\left(\frac{f(a,b+s) - f(a,b)}{s}\right)}^{f_y(a,b)}}{h}$$

Set $g(h,s)$ = expression in the limit. Then, if $\lim\limits_{(h,s)\to(0,0)} g(h,s)$ exists then the two quantities are equal (2 paths $(h,s) \to (0,s) \to (0,0)$ vs $(h,s) \to (h,0) \to (0,0)$)

follows from cont of f_{xy} & f_{yx}.

Example: [Continuity can be checked with this Test]

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0). \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Can check: f is continuous at $(0,0)$

$f_x(0,0) = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$, $f_y(0,0) = 0$.

$$f_x(x,y) = \frac{(x^2+y^2)(3x^2y - y^3) - (x^3y - xy^3)2x}{(x^2+y^2)^2} = \frac{3x^4y - x^2y^3 + 3x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2}$$

for $(x,y)\neq(0,0)$

$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

Similarly, $f_y(x,y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$. for $(x,y)\neq(0,0)$.

$$f_{xy}(0,0) = \lim_{y\to 0} \frac{f_x(0,y) - f_x(0,0)}{y} = \lim_{y\to 0} \frac{\frac{-y^5}{y^4} - 0}{y} = \lim_{y\to 0} \frac{-y^5}{y^5} = \boxed{-1}$$

$$f_{yx}(0,0) = \lim_{x\to 0} \frac{f_y(x,0) - f_y(0,0)}{x} = \lim_{x\to 0} \frac{\frac{x^5}{x^4} - 0}{x} = \boxed{1}$$

different values!

Reason: $\lim\limits_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist! Also $\lim\limits_{(x,y)\to(0,0)} f_{yx}(x,y)$ doesn't exist (symmetry!)

$$f_{xy}(x,y) = \frac{(x^2+y^2)^2(4x^3y + 8xy^3) - (x^4y + 4x^2y^3 - y^5)\,2(x^2+y^2)2x}{(x^2+y^2)^4}$$

§4 Differentiability:

In the one variable case: $f:\mathbb{R}\to\mathbb{R}$ is differentiable at $x=a$ if $f'(a)$ exists. Furthermore the graph of f would be smooth at $(a,f(a))$ and a unique tangent line at $(a,f(a))$, of slope $f'(a)$.

[In the 2 variables case: replace tangent line with tangent plane (§13.7)].

Formally:
$$\underset{(\Delta x)}{\varepsilon} = \underbrace{\frac{f(a+\Delta x)-f(a)}{\Delta x}}_{\text{slope of secant line}} - \underbrace{f'(a)}_{\text{slope of tangent line}} \xrightarrow[\Delta x\to 0]{} 0$$

Rewrite:
$$\varepsilon\Delta x = \underbrace{f(a+\Delta x)-f(a)}_{=\Delta y} - f'(a)\Delta x.$$

$$f(a+\Delta x)-f(a) = \Delta y = f'(a)\Delta x + \varepsilon\Delta x \quad \text{and} \quad \varepsilon \xrightarrow[\Delta x\to 0]{} 0 \quad (\text{"linear approximation"})$$

If f is differentiable, then the change in f between a and a nearby pt $a+\Delta x$ is represented by $f'(a)\Delta x$ plus a quantity $\varepsilon\Delta x$ where $\lim_{\Delta x\to 0}\varepsilon = 0$.

Def: The function $z=f(x,y)$ is differentiable at (a,b) provided $f_x(a,b)$ & $f_y(a,b)$ exist and the change $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ equals
$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$
where for fixed a,b, ε_1 & ε_2 are functions depending ONLY on Δx & Δy with
$$(\varepsilon_1,\varepsilon_2) \xrightarrow[(\Delta x,\Delta y)\to(0,0)]{} 0.$$

A function is differentiable on an open set R if it's differentiable at every pt of R.

Theorem: If f_x & f_y are continuous at (a,b), then f is differentiable at (a,b)

Remark: Existence of f_x & f_y is NOT enough!

Example:
$$f(x,y) = \begin{cases} \dfrac{3xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

Approach along lines: $y=mx$ $\quad \displaystyle\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} \frac{3xy}{x^2+y^2} = \frac{3m}{1+m^2}$. value varies with m. so f is not continuous at $(0,0)$ (*)

$$\begin{cases} f_x(0,0) = \lim_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\dfrac{0-0}{h} = 0 \\ f_y(0,0) = \lim_{h\to 0}\dfrac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\dfrac{0-0}{h} = 0\end{cases}$$

$$f_x(x,y) = \frac{3y(x^2+y^2)-3xy\,2x}{(x^2+y^2)^2} = \frac{3y(x^2-y^2)}{(x^2+y^2)^2}$$ $(x,y)\neq(0,0)$ is NOT continuous at $(0,0)$

Again: approach along lines $y = mx$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f_x(x,y) = \lim_{x\to 0} \frac{3mx\left(x^2 - m^2x^2\right)}{\left(x^2+(mx)^2\right)^2} = \lim_{x\to 0} \frac{3x^3\, m\left(1-m^2\right)}{x^4\left(1+m^2\right)^2} = \begin{cases} \cancel{+\infty} & |m|<1 \\ 0 & |m|=1 \\ -\infty & |m| \end{cases}$$

The value is $\pm\infty$ or 0 depending on m

The limit is $\begin{cases} +\infty & m<-1 \\ 0 & m=-1 \\ -\infty & -1<m<0 \\ 0 & m=0 \\ +\infty & 0<m<1 \\ 0 & m=1 \\ -\infty & m>1 \end{cases}$

The next result shows that (*) implies that f is not differentiable at $(0,0)$

Theorem: f differentiable at (a,b), then f is continuous at (a,b).

Proof By definition $\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$.

But $(\text{RHS}) \xrightarrow[(\Delta x,\Delta y)\to(0,0)]{} 0$ Thus, $f(x+\Delta x, y+\Delta y) \xrightarrow[(\Delta x,\Delta y)\to(0,0)]{} f(a,b)$, or equivalently,

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(a,b)$$