Lecture XVI (2/22/16) §13.4 (cont) & 13.5: The Chain Rule

§1 Differentiability:

Recall: • The function $z = f(x,y)$ is differentiable at (a,b) if:

1) $f_x(a,b)$ & $f_y(a,b)$ both exist

2) The change $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ equals

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

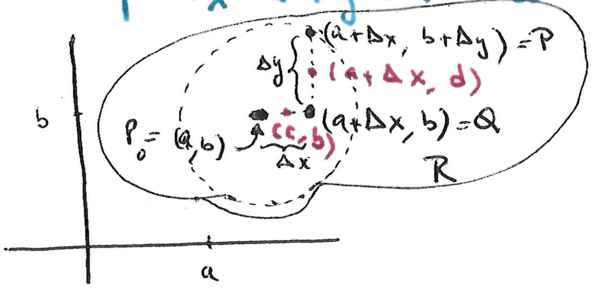
where $\varepsilon_1, \varepsilon_2$ are functions only depending on $a, b,$ Δx & Δy, and $(\varepsilon_1, \varepsilon_2) \xrightarrow[(\Delta x, \Delta y)\to(0,0)]{} (0,0)$.

• f is differentiable on an open set R if it's differentiable at every pt of R.

Write: $df := \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$ = differential of f.

Theorem 1: If f_x & f_y are continuous at (a,b), then f is differentiable at (a,b)

Proof:



The region R contains the three points
$$\begin{cases} P_0 = (a,b) \\ Q = (a+\Delta x, b) \\ P = (a+\Delta x, b+\Delta y) \end{cases}$$

We want to show that $\Delta z = f(P) - f(P_0) \overset{?}{=} f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

Where $\varepsilon_1, \varepsilon_2$ depend only on $a, b, \Delta x$ & Δy, and $(\varepsilon_1, \varepsilon_2) \xrightarrow[(\Delta x, \Delta y)\to(0,0)]{} (0,0)$

We view the change Δz as $\underbrace{(f(P) - f(Q))}_{=:\,\Delta z_1} + \underbrace{(f(Q) - f(P_0))}_{=:\,\Delta z_2}$.

• $\Delta z_1 = f(a+\Delta x, \underline{\underline{b}}) - f(a, \underline{\underline{b}}) = f_x(c, \underline{\underline{b}})\,\Delta x$ for some c between a & $\Delta x + a$ using the Mean Value Thm for the real-valued function $g(x) := f(x, \underline{\underline{b}})$ defined around $x = a$.

Similarly:

• $\Delta z_2 = f(\underline{a+\Delta x}, b+\Delta y) - f(\underline{a+\Delta x}, b) = f_y(\underline{a+\Delta x}, d)\,\Delta y$ for some d between b & $b+\Delta y$ using the Mean Value Thm for the real-valued function $h(y) = f(\underline{a+\Delta x}, y)$ defined around $y = b$ [Notice that the choice of d varies with Δx]

Then $\Delta z = \Delta z_1 + \Delta z_2 = f_x(c,b)\,\Delta x + f_y(a+\Delta x, d)\,\Delta y$

$$= \underbrace{(f_x(c,b) - f_x(a,b))}_{=:\,\varepsilon_1}\Delta x + \underbrace{(f_y(a+\Delta x, d) - f_y(a,b))}_{=:\,\varepsilon_2}\Delta y + f_x(a,b)\,\Delta x + f_y(a,b)$$

sum & subtract $f_x(a,b)$ & $f_y(a,b)$

• As $\Delta x \to 0$ we have $c \to a$ and as $\Delta y \to 0$, we have $d \to b$.

Since f_x & f_y are continuous at (a,b), the expressions

$$\begin{cases} \varepsilon_1 := f_x(c,b) - f_x(a,b) \xrightarrow[\Delta x \to 0]{} f_x(a,b) - f_x(a,b) = 0 \\ \quad \hookrightarrow \text{depends only on } a, b, \Delta x, \Delta y \\ \varepsilon_2 := f_y(a+\Delta x, d) - f_y(a,b) \xrightarrow[(\Delta x, \Delta y) \to (0,0)]{} f_y(a,b) - f_y(a,b) = 0 \\ \quad \hookrightarrow \text{depends only on } a, b, \Delta x, \Delta y. \end{cases}$$

So f is differentiable at (a,b) as we wanted to show.

Remark: The existence of f_x & f_y does not guarantee differentiability!

Example $f(x,y) = \begin{cases} \dfrac{3xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

We show f_x & f_y exist by using definition:

$$\begin{cases} f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0-0}{h} = \boxed{0} \\ f_y(0,0) = \lim\limits_{h \to 0} \dfrac{f(0,h) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0-0}{h} = \boxed{0} \end{cases}$$

For $(x,y) \neq (0,0)$ we use the derivation rules:

$$f_x(x,y) = \frac{3y(x^2+y^2) - 3xy(2x)}{(x^2+y^2)^2} = \frac{3y(x^2-y^2)}{(x^2+y^2)^2} \quad \text{for } (x,y) \neq (0,0)$$

- f_x is not continuous at $(0,0)$:

  Limit along lines $y = mx$: $\lim\limits_{x \to 0} \dfrac{3mx(x^2 - m^2x^2)}{(x^2 + m^2x^2)^2} = \lim\limits_{x \to 0} \dfrac{3mx^3(1-m^2)}{x^4(1+m^2)}$ does not exist!

  (Eg: $m = \frac{1}{2}$: $\lim\limits_{x \to 0^+} = +\infty$, $\lim\limits_{x \to 0^-} = -\infty$)

- f is not continuous at $(0,0)$

  Approach along lines $y = mx$, $m \neq 0$: $\lim\limits_{x \to 0} \dfrac{3xm\cancel{x}}{x^2 + m^2\cancel{x}^2} = \lim\limits_{x \to 0} \dfrac{3m}{1+m^2} = \dfrac{3m}{1+m^2}$

  the value of the limit varies w/ m, so the limit does not exist!

The next result shows that a discontinuous function is not differentiable.

Theorem 2: If f is differentiable at (a,b), then f is continuous at (a,b).

Proof: By definition of differentiability, we see:

$$f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta z \xrightarrow[(\Delta x, \Delta y) \to (0,0)]{} 0$$

(RHS in the def. has limit $= 0$)

so $f(a,b) = \lim\limits_{(\Delta x, \Delta y) \to (0,0)} f(a+\Delta x, b+\Delta y) = \lim\limits_{(x,y) \to (a,b)} f(x,y)$.

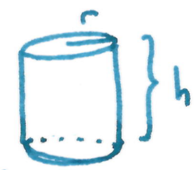
- Can use differentials to estimate effect of measurement errors

Examples ① If a radius of a sphere is 2 with an error of ±0,01 units, find the error in measuring its volume:

$$V_{(r)} = \frac{4}{3}\pi r^3 \rightsquigarrow dV = 4\pi r^2 \underbrace{dr}_{=\pm 0.01} = \pm 4\pi\, 2^2\,(0.01) = \boxed{\pm 0.16\pi}$$

② Find the maximum error in computing the volume of a cylinder if the errors in computing the radius and height are 5% and 2% respectively.

5% means $\frac{dr}{r} = 5$; 2% means $\frac{dh}{h} = 2$.



$$V(r,h) = \pi r^2 h \Rightarrow dV = 2\pi r h\, dr + \pi r^2 dh$$

so $\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h} = 10+2 = 12 \Rightarrow$ error in computing the volume is 12%.

§13.5 Chain Rule:

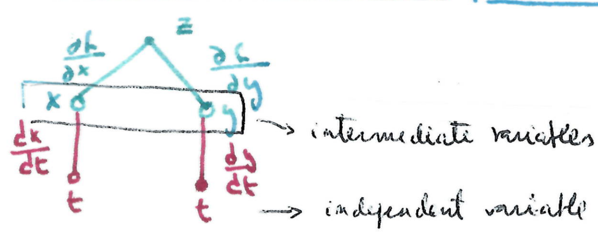
Recall (1 variable): $f, g : \mathbb{R} \longrightarrow \mathbb{R} \rightsquigarrow f\circ g : \mathbb{R} \longrightarrow \mathbb{R}$ $\quad f\circ g_{(t)} = f(g(t))$ composition

Then, assuming g differentiable at $t=a$ & f differentiable at $g(a)=b$, then $f\circ g$ is differentiable at $t=a$ and $(f\circ g)'(a) = f'(g(a))\cdot g'(a)$.

Q: What to do if $z = f(x,y)$ and $x = x(t)$, $y = y(t)$?
$[\,x = x(t_1, \dots, t_n) \quad y = y(t_1, \dots, t_n)\,]$

§1 The Chain Rule with One Independent Variable?

Idea:



[Tree dependency]

Theorem: Let $f : D \longrightarrow \mathbb{R}$ differentiable at (b,c) and $(x,y): \mathbb{R} \longrightarrow D$ differentiable at $t=a$, where $x(a) = b$ & $y(a) = c$. Then $f\circ(x,y) = f(x_{(t)}, y_{(t)})$ is differentiable at $t=a$ and

$$\frac{d\, f(x(t), y(t))}{dt}(a) = \frac{\partial f}{\partial x}(b,c)\,\frac{dx}{dt}(a) + \frac{\partial f}{\partial y}(b,c)\,\frac{dy}{dt}(a).$$

For general functions $g_{(t)} = f(x_1(t), \dots, x_n(t)) \rightsquigarrow$

$$\frac{dg}{dt}(a) = \frac{\partial f}{\partial x_1}(x_1(a), \dots, x_n(a))\,\frac{dx_1}{dt}(a) + \dots + \frac{\partial f}{\partial x_n}(x_1(a), \dots, x_n(a))\,\frac{dx_n}{dt}(a).$$

(n summands).

- In short, $\frac{df}{dt}$ = sum of the contributions from each branch of the tree

Eg: $$\frac{df(x(t), y(t))}{dt}(a) = \underbrace{\frac{\partial f}{\partial x}(b,c)\,\frac{dx}{dt}(a)}_{\text{left branch}} + \underbrace{\frac{\partial f}{\partial y}(b,c)\cdot\frac{dy}{dt}(a)}_{\text{right branch}}.$$

Proof: $\Delta x = x(t+\Delta t) - x(t)$
$\Delta y = y(t+\Delta t) - y(t)$

Assume $(x(t), y(t))$ lies in D when t lies in between a & $a+\Delta t$.

Since f is differentiable at (b,c), then

$$\Delta z = \frac{\partial f}{\partial x}(a,b)\,\Delta x + \frac{\partial f}{\partial y}(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Divide by Δt & take limit as $\Delta t \to 0$.

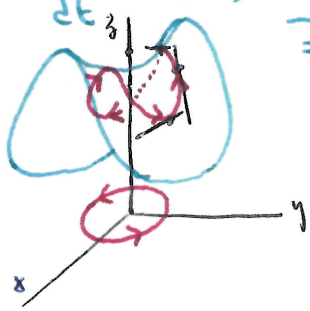
$$\underset{\substack{\downarrow\,\Delta t\to 0\\ \frac{d(f(x(t),y(t)))}{dt}(a)}}{\frac{\Delta z}{\Delta t}} = \frac{\partial f}{\partial x}(a,b)\underset{\substack{\downarrow\,\Delta t\to 0\\ \frac{dx}{dt}(a)}}{\frac{\Delta x}{\Delta t}} + \frac{\partial f}{\partial y}(a,b)\underset{\substack{\downarrow\,\Delta t\to 0\\ \frac{dy}{dt}(a)}}{\frac{\Delta y}{\Delta t}} + \underset{\substack{\downarrow\\ 0\\ (*)}}{\varepsilon_1}\underset{\substack{\downarrow\,\Delta t\to 0\\ \frac{dx}{dt}(a)}}{\frac{\Delta x}{\Delta t}} + \underset{\substack{\downarrow\\ 0\\ (*)}}{\varepsilon_2}\underset{\substack{\downarrow\,\Delta t\to 0\\ \frac{dy}{dt}(a)}}{\frac{\Delta y}{\Delta t}}$$

(*) Note: $(\varepsilon_1, \varepsilon_2) \xrightarrow[(\Delta x,\Delta y)\to(0,0)]{} (0,0)$ and $(\Delta x, \Delta y) \xrightarrow[\Delta t\to 0]{} (0,0)$, so by continuity of $\Delta x, \Delta y$ at $t=a$ we conclude that $(\varepsilon_1, \varepsilon_2) \xrightarrow[\Delta t\to 0]{} (0,0)$.

⊢

Example: $z = f(x,y) = x^2 - 3y^2 + 20$ (hyperbolic paraboloid) where $x = \cos t$, $y = \sin t$.

Then $\frac{dz}{dt} = (2x)\underbrace{x'(t)}_{=-\sin t} + (-6y)\underbrace{y'(t)}_{=\cos t} = -2\cos t\sin t - 6\sin t\cos t$ (by the chain rule) $= \boxed{-8\cos t\sin t}$



Check: $z = z(t) = \cos^2 t - 3\sin^2 t + 20 \Rightarrow \frac{dz}{dt} = -2\cos t\sin t - 3\cdot 2\sin t\cos t = \boxed{-8\cos t\sin t}$

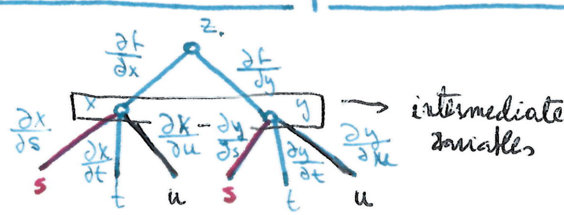
$\frac{dz}{dt}\Big|_{t=a}$ = rate of change of the elevation z with respect to t when walking along the path on the surface.

$\begin{cases}\text{positive rate = going up}\\ \text{negative " = going down}\end{cases}$

§2 The chain rule with several independent variables:

Dependence tree

Example:



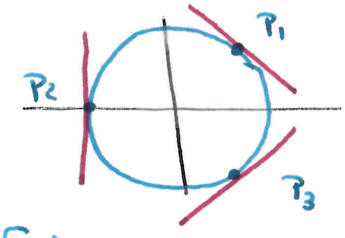
$\underset{(s,t,u)}{z} = f(x(s,t,u), y(s,t,u))$

Theorem: $f: \mathbb{R}^r \to \mathbb{R}$ differentiable at $\underline{b} = (b_1, \dots, b_r)$ and $x_1, \dots, x_r : D \to \mathbb{R}$ where each x_i is differentiable at $\underline{p}$ in D $(p_1,\dots,p_n)$ and $x_i(\underline{p}) = b_i$, then $g(t_1,\dots,t_n) = f(x_1(t_1,\dots,t_n), \dots, x_r(t_1,\dots,t_n))$ is differentiable at $\underline{p}$ and

$$\frac{\partial g}{\partial t_i} = \frac{\partial f}{\partial x_1}(\underline{b})\frac{\partial x_1}{\partial t_i}(\underline{p}) + \frac{\partial f}{\partial x_2}(\underline{b})\frac{\partial x_2}{\partial t_i}(\underline{p}) + \dots + \frac{\partial f}{\partial x_r}(\underline{b})\frac{\partial x_r}{\partial t_i}(\underline{p})$$ for all i.

§3 Implicit differentiation:

Example:



$x^2+y^2=1$ GOAL: Find the tangent lines to the circle at the 3 points P_1, P_2 & P_3.

$P_1=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$, $P_2=(-1,0)$, $P_3=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

Solution 1: Express the circle as a parametric curve:

$\vec{r}(t)=\langle \cos t, \sin t\rangle \quad 0\le t\le 2\pi \quad \rightsquigarrow \vec{r}\,'(t)=\langle -\sin t, \cos t\rangle$

$P_1=\vec{r}\left(\frac{\pi}{4}\right)$, $P_2=\vec{r}(\pi)$, $P_3=\vec{r}\left(\frac{7}{4}\pi\right)$

tangent directions: at $P_1=\vec{r}\,'\left(\frac{\pi}{4}\right)=\langle-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle$ at $P_2=\vec{r}\,'(\pi)=\langle 0,-1\rangle$, at $P_3=\vec{r}\,'\left(\frac{7\pi}{4}\right)=\langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle$

$\Rightarrow \ell_{P_1}(s)=\langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle+s\langle-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle$ and similarly for P_2 & P_3.

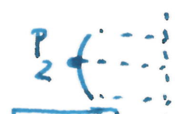
In general: Tangent line at $P=\vec{r}(t_0)$ is $\boxed{\ell_P(s)=\vec{r}(t_0)+s\,\vec{r}\,'(t_0)}$

Solution 2: If we cannot find a parametrization, near each point we can try to express $y=y(x)$ or $x=x(y)$ from the equation $(x^2+y^2=1)$

Near P_1: $y=\sqrt{1-x^2}$

Near P_3: $y=-\sqrt{1-x^2}$

Near P_2: cannot write y as a function of x, but can write x as a function of y $\quad x=x(y) \quad x=-\sqrt{1-y^2}$



- For P_1 & P_3: equation of the tangent line through (a,b) is

$$\boxed{y-b=y'(a)\,(x-a)}$$

For P_1: $y'(x)=\frac{-x}{\sqrt{1-x^2}}$

For P_3: $y'(x)=\frac{x}{\sqrt{1-x^2}}$.

- For P_2: equation of the tangent line through $(-1,0)$ is

$$x-(-1)=x'(0)\,(y-0)$$

$x'(0)=\left.\frac{y}{\sqrt{1-y^2}}\right|_{y=0}=0$.

$\Rightarrow$ Tangent line: $x+1=0$.

Next Time: Alternative method if we cannot explicitly write $y=y(x)$ or $x=x(y)$