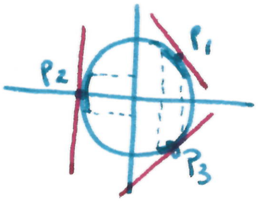
Lecture XVII (2/24/16) §13.5 Implicit differentiation & §13.6: Directional derivatives & the gradient

§13.4 Implicit differentiation:

Example (Last Time) Find the tangent lines to the curve $x^2+y^2=1$ at 3 points



$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$P_3 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$P_2 = (-1, 0)$

- **Soln 1**: Write the curve in parametric form $\vec{r}(t) = \langle \cos t, \sin t\rangle$ $0 \leq t \leq 2\pi$. For each P find t_0 where $P = \vec{r}(t_0)$. Then the tangent line has vector equation
$$\ell(s) = \vec{r}(t_0) + s\, \vec{r}\,'(t_0).$$

- **Soln 2**: Locally around each point we can write $y = y(x)$ (y as a function of x) ① OR $x = x(y)$ (x as a function of y) ②

The tangent line has an equation in $\mathbb{R}^2$:

- For ①: (x,y) in the tangent line through $P=(a,b)$ if and only if:
$$y - b = y'(a)\,(x-a)$$
$\left[\text{For } P_1: \ y - \frac{\sqrt{2}}{2} = \underbrace{\boxed{\frac{-\frac{\sqrt{2}}{2}}{\sqrt{1/2}}}}_{=-1}\left(x - \frac{\sqrt{2}}{2}\right) \quad \text{For } P_3: \ y + \frac{\sqrt{2}}{2} = \left(x - \frac{\sqrt{2}}{2}\right).\right]$

- For ②: $x - a = x'(b)\,(y-b)$ $\quad [\text{For } P_2: x + 1 = 0\,(y-0)]$

- **Soln 3**: We don't need to know the expression of $\begin{cases} y = y(x) \\ x = x(y) \end{cases}$ to compute $\begin{matrix} y'(a). \\ x'(b) \end{matrix}$

Write $\underbrace{F(x,y)}_{} = x^2+y^2 = \underbrace{0}_{}$. Assume $y = y(x)$ and differentiate both sides with respect to x, using the Chain Rule:
$$0 = \frac{d}{dx}\left(F(x, y(x))\right) = F_x(x,y)\cdot \underbrace{\frac{dx}{dx}}_{=1} + F_y(x,y)\cdot \boxed{\frac{dy}{dx}}$$
↪ what we want!

Provided $F_y \neq 0$ (in our example: $F_y = 2y \neq 0$ for P_1 & P_3), we get $\frac{dy}{dx} = \frac{-F_x}{F_y}$.

$\left(\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}\right.$ in our example$\left.\right)$.

⇝ The tangent line at (a,b) has equation $(y-b) = \boxed{\frac{-F_x(a,b)}{F_y(a,b)}}\,(x-a)$.

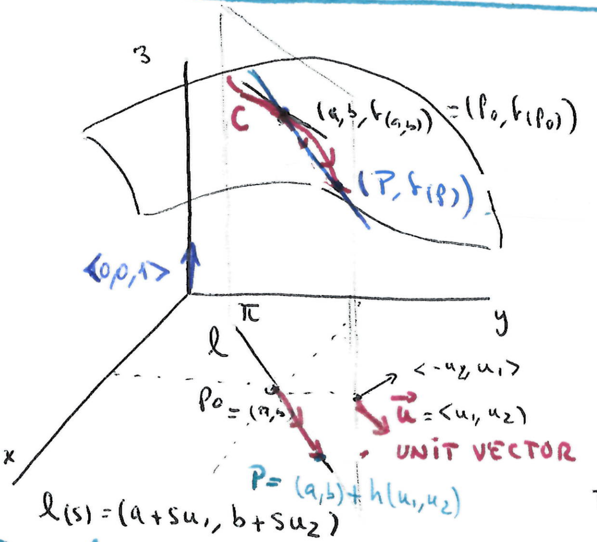
↪ slope

Theorem: Assume $F(x,y)$ is differentiable at (a,b) where $F(a,b)=0$. Assume locally around (a,b) the level curve $F(x,y)=0$ defines y as a differentiable function of x. Provided $F_y(a,b) \neq 0$, we conclude
$$\frac{dy}{dx}(a) = \frac{-F_x(a,b)}{F_y(a,b)}.$$

Note: If F_x & F_y are continuous in a ball around (a,b), then $F_y(a,b) \neq 0$ ensures that $y = y(x)$ locally around (a,b) & the previous theorem ensures $\dfrac{dy}{dx}(a) = \dfrac{-F_x(a,b)}{F_y(a,b)}$
[Implicit Function Theorem]

Application: Speed of fluid flows [Recitation 6].

§13.6 Directional derivatives and the gradient:



GOAL: Understand the rate of change of $f: D \to \mathbb{R}$ differentiable at (a,b) when we approach (a,b) in an arbitrary direction $\vec{u}$ in $\mathbb{R}^2$ where $\vec{u}$ is a UNIT vector

Notation: $D_{\vec{u}}(f)(a,b)$

Examples: • $u = \langle 1,0\rangle \rightsquigarrow D_{\vec{u}}(f)_{(a,b)} = f_x(a,b)$

• $u = \langle 0,1\rangle \rightsquigarrow D_{\vec{u}}(f)_{(a,b)} = f_y(a,b)$

π = plane through P_0 with directions $(\vec{u},0)$ & $(0,0,1)$ $\Rightarrow$ $\vec{\eta} = \langle u_1, u_2, 0\rangle \times \langle 0,0,1\rangle = \langle u_2, -u_1, 0\rangle$ normal direction
Eqn: $x u_2 - u_1 y = a u_2 - b u_1$

The slope of the secant line between $(P_0, f(P_0))$ & $(P, f(P))$ where $P_0 = (a,b)$, $P = (a+hu_1, b+hu_2)$

is $\dfrac{f(a+hu_1, b+hu_2) - f(a,b)}{h} \xrightarrow[h\to 0]{}$ slope of the tangent line to the curve C cut out by the graph of f & the plane π

Definition Fix $\vec{u} = (u_1, u_2)$ unit vector & $f: D \to \mathbb{R}$ differentiable at (a,b) in D.
The directional derivative of f at (a,b) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

whenever this limit exists.

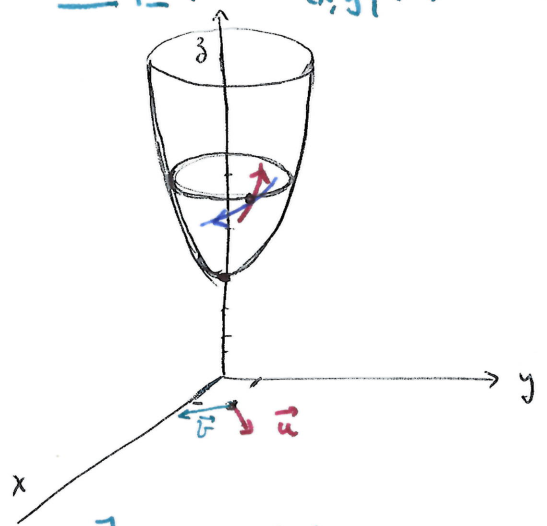
Key: The curve C is the graph of the function $g(s) = f(\overbrace{a+su_1}^{x=x(s)}, \overbrace{b+su_2}^{y=y(s)})$ drawn in the plane π. By definition: $D_{\vec{u}} f(a,b) = \lim_{h\to 0} \dfrac{g(h) - g(0)}{h} = g'(0)$

By the chain rule:
$$g'(0) = f_x(a,b) \cdot (a+su_1)'\Big|_{s=0} + f_y(a,b)\cdot(b+su_2)'\Big|_{s=0}$$
$$= f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2$$
$$= \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle$$

Definition: The gradient of f at (a,b) is $\nabla f_{(a,b)} = \langle f_x(a,b), f_y(a,b)\rangle$

Theorem: If $\vec{u}$ is a unit vector and f is differentiable at (a,b), then:
$$\boxed{D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}} \quad (*)$$

Example: $z = f(x,y) = 4 + x^2 + 3y^2$. Its graph is an elliptic paraboloid



$P_0 = (1,1) \rightsquigarrow f(P_0) = 8$

$\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$

$\nabla f_{(x,y)} = \langle 2x, 6y \rangle \rightsquigarrow \nabla f_{(1,1)} = \langle 2, 6 \rangle$

$D_{\vec{u}} f_{(P_0)} = \langle 2,6 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{8}{\sqrt{2}} > 0$ (increases height)

$D_{\vec{v}} f_{(P_0)} = \langle 2,6 \rangle \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle = 1 - 3\sqrt{3} < 0$ (descends)

§2 Meaning of the gradient:

Assume $\nabla f_{(a,b)} \neq \vec{0}$.

Using (*) & the geometric interpretation of the dot product:

$$D_{\vec{u}} f_{(a,b)} = \nabla f_{(a,b)} \cdot \vec{u} = \underbrace{|\nabla f_{(a,b)}|}_{\neq 0} \underbrace{|\vec{u}|}_{=1} \cos(\theta)$$

θ = angle between the vectors $\nabla f_{(a,b)}$ & $\vec{u}$. $(0 \leq \theta \leq \pi)$

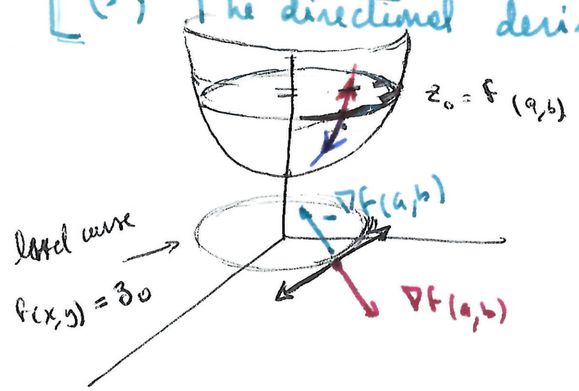
So
- $D_{\vec{u}} f_{(a,b)}$ is MAXIMAL when $\cos\theta = 1 \iff \theta = 0 \iff \vec{u}$ & $\nabla f_{(a,b)}$ have the SAME direction
- ——— MINIMAL ——— $\cos\theta = -1 \iff \theta = \pi \iff \vec{u}$ & $\nabla f_{(a,b)}$ have OPPOSITE direction
- $D_{\vec{u}} f_{(a,b)}$ has value 0 when $\cos\theta = 0 \iff \vec{u}$ & $\nabla f_{(a,b)}$ are orthogonal.

Thm 1 f differentiable at (a,b) & $\nabla f_{(a,b)} \neq \vec{0}$. Then

(1) f has its maximum rate of increase at (a,b) in the direction of $\nabla f_{(a,b)}$. $\left(\vec{u} = \frac{\nabla f_{(a,b)}}{|\nabla f_{(a,b)}|}\right)$ & the rate of change in this direction is $|\nabla f_{(a,b)}|$ (STEEPEST ASCENT DIRECTION = $\nabla f_{(a,b)}$ direction)

(2) f has its maximum rate of decrease at (a,b) in the direction of $-\nabla f_{(a,b)}$ $\left(\vec{u} = \frac{-\nabla f_{(a,b)}}{|\nabla f_{(a,b)}|}\right)$ & the rate of change in this direction is $-|\nabla f_{(a,b)}|$ (STEEPEST DESCENT DIRECTION = $-\nabla f_{(a,b)}$ direction)

(3) The directional derivative is 0 in any direction orthogonal to $\nabla f_{(a,b)}$.



Recall: $f(x,y) = z_0$ is a level curve (here: $f_{(a,b)} = z_0$)

Thm 2: If f is differentiable at (a,b), the tangent line to the level curve $f(x,y) = z_0$ is orthogonal to $\nabla f_{(a,b)}$ (if $\nabla f_{(a,b)} \neq \vec{0}$)