Lecture XVIII (2/26/16) §13.6: Gradient & level curves, §13.7: Tangent planes & linear approximation

§13.6 The gradient

$\boxed{\text{Assume all our functions are differentiable}}$

- Let $f(x,y)$ be a function of 2 variables:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \text{ is the gradient of } f.$$

It's a vector valued function of 2 variables

- Similarly, for a function of 3 variables $f(x,y,z)$

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$$

Example: $f = 3x^2y + yz^2 + y^3 \rightsquigarrow \nabla f = \langle 6xy, 3x^2+z^2+3y^2, 2yz \rangle$

Properties: (functions of 2 or more variables)

(1) $D_{\vec{u}} f_{(x_0,y_0)} = \nabla f_{(x_0,y_0)} \cdot \vec{u}$ for every unit vector $\vec{u}$.

↳ rate of change of f at (x_0,y_0) in the direction of $\vec{u}$.

(2) $D_{\vec{u}} f_{(x_0,y_0)} = |\nabla f_{(x_0,y_0)}| \cos\theta$ $\quad\theta$ = angle between $\nabla f(x_0,y_0)$ and $\vec{u}$.

is maximum when $\theta = 0$

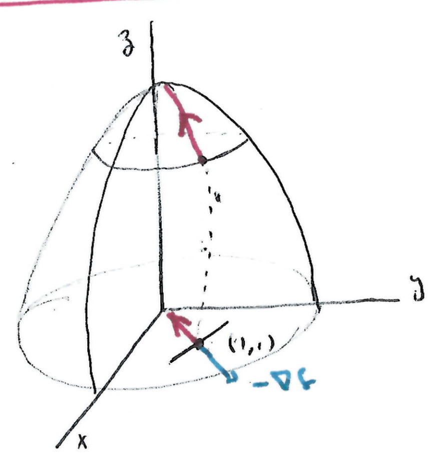
$\boxed{\nabla f_{(x_0,y_0)} \text{ points in the direction of largest } \underline{\text{increase}} \text{ of } f \text{ at } (x_0,y_0)}$

Example $f(x,y) = 9 - x^2 - y^2 \qquad P_0 = (1,1)$

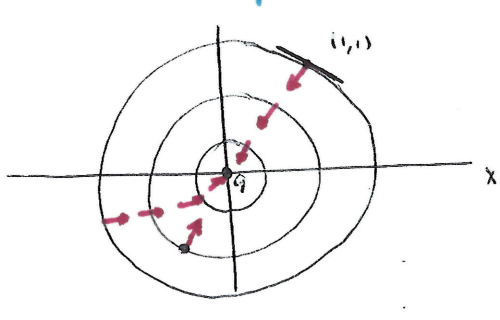
$\nabla f = \langle -2x, -2y \rangle$

$\nabla f_{(1,1)} = \langle -2, -2 \rangle$

$\rightsquigarrow \vec{u} = \langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$ is the direction of greatest increase of f.



- Contour map of level curves



$\nabla f_{(-1,-2)} = \langle 2,4 \rangle$

$\rightsquigarrow \vec{u} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

(3) $D_{\vec{u}} f$ is minimum when $\theta = \pi$ $(180°)$

$\boxed{-\nabla f_{(x_0, y_0)} \text{ points in the direction of fastest } \underline{\text{decrease}} \text{ of } f \text{ at } (x_0, y_0)}$

[See Exercise 6 in Recitation 6 Handout]

(4) $\begin{cases} \nabla f_{(x_0,y_0)} & \text{is perpendicular to the level curve of } f \text{ at } (x_0,y_0) \\ \nabla f_{(x_0,y_0,z_0)} & \text{———————— surface of } f \text{ " } (x_0,y_0,z_0) \end{cases}$

(when $\nabla f \neq \vec{0}$)

[it's perpendicular to any curve in the level surface passing through (x_0, y_0, z_0)]

Pf/ • For level curves: we pick a parameterization of the level curve $f(x,y) = c$ near (x_0, y_0)

$\vec{r}(t) = \langle x(t), y(t) \rangle$ where $\vec{r}(t_0) = \langle x_0, y_0 \rangle$

We know $f(x(t), y(t)) = c$ is constant for all t.

Differentiate with respect to t using the Chain Rule

$$\begin{array}{ccc} & f & \\ x & & y \\ | & & | \\ t & & t \end{array}$$

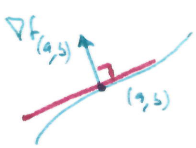
$$0 = \frac{d}{dt} c = f_x(x_0,y_0)\, x'(t_0) + f_y(x_0,y_0)\, y'(t_0) = \nabla f_{(x_0,y_0)} \cdot \underbrace{\vec{r}\,'(t_0)}_{\text{tangent direction at } (x_0,y_0)}$$

• For level surfaces: $f(x_0, y_0, z_0) = c$, $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ parametric curve lying on $f(x,y,z) = k$ & passing through $(x_0, y_0, z_0) = \vec{r}(t_0)$.

$f(x(t), y(t), z(t)) = k$ constant

We take derivative wrt t and get $0 = \nabla f_{(x_0,y_0,z_0)} \cdot \underline{\vec{r}\,'(t_0)}$

Consequences: (1) The equation of the tangent line to the level curve $f(x,y) = c$ at (a,b)

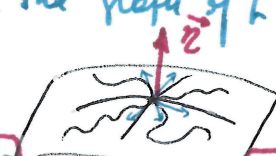


$$\boxed{f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b) = 0} \quad (\text{line in } \mathbb{R}^2)$$

(2) If the surface is the graph of $z = f(x,y)$, then it's the level 0 surface of the function $F(x,y,z) = z - f(x,y)$ $\rightsquigarrow$ $\nabla F_{(x,y,z)} = \langle -f_x, -f_y, 1 \rangle = \vec{z}(x,y)$

. The vector $\vec{z}(x,y)$ is perpendicular to the tangent line to any curve on the graph of f through $(x, y, f(x,y))$

so $\vec{r}(t) = \langle x(t), y(t), f(x(t), y(t)) \rangle$



$\boxed{\text{The plane through } (x,y,f(x,y)) \text{ with normal } \vec{z}(x,y) \text{ contains the tangent vectors to } \underline{\text{all}} \text{ of the parametric curves } \vec{r}(t) \text{ above } \rightsquigarrow \text{ tangent plane!}}$

§ 13.7. Tangent planes and linear approximation

Definition: The tangent plane to a surface $F(x,y,z)=0$ at $(a,b,c)=P_0$ is given by $\boxed{F_x(P_0)(x-a)+F_y(P_0)(y-b)+F_z(P_0)(z-c)=0}$

(Normal vector: $\vec{n}(P_0)=\nabla F(P_0)$)

• Special case: $F = z-f(x,y)$; then $\vec{n}=\langle -f_x,-f_y,1\rangle$ and the eqn becomes

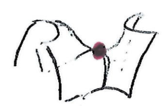
$$-f_{x\,(x_0,y_0)}(x-a)-f_{y\,(x_0,y_0)}(y-b)+(z-f(a,b))=0$$

Equivalently: (*) $\boxed{z=f(a,b)+f_{x\,(a,b)}(x-a)+f_{y\,(a,b)}(y-b)}$

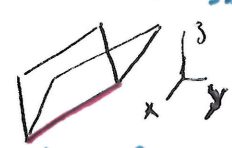
Def: $\vec{n}$ is called the NORMAL VECTOR to the surface.

Note: ① If f is not differentiable at (a,b), then there is no tangent plane at $(a,b,f(a,b))$

Eg: Hyperbolic paraboloid $z=x^2-y^2$, $\nabla f(0,0)=\vec{0}$ or $\nabla f_{(a,b)}=(0,0)$



; $z=|y|$, & pts $(x,0)$



no $f_y(x,0)$.

Sufficient condition: $\nabla f_{(x,y)}$ continuous at (a,b) & $\nabla f_{(a,b)}\neq\vec{0}$.

② The (RHS) of (*) is the graph of a linear function:

$$\boxed{L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)}$$

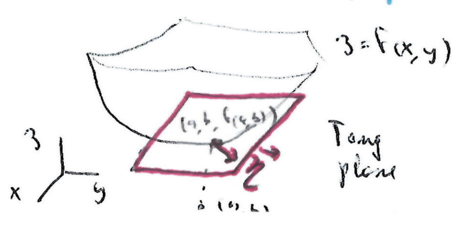
→ $L(x,y)$ is the UNIQUE linear function such that:

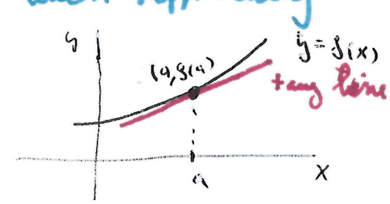
$$\begin{cases} L(a,b)=f(a,b) \\ \dfrac{\partial L}{\partial x}(a,b)=\dfrac{\partial f}{\partial x}(a,b) \quad \& \quad \dfrac{\partial L}{\partial y}(a,b)=\dfrac{\partial f}{\partial y}(a,b) \end{cases}$$

So $f(x,y) \underset{\text{(approx.)}}{\approx} L(x,y)$ near the point (a,b) (very close!) $[L(x,y)-f(a,b)=df_{(a,b)}]$ $\{\Delta x = x-a,\ \Delta y = y-b$

Def $L(x,y)$ is called the linear approximation of $f(x,y)$ at (a,b)

Note: The graph of f coincides with the graph of L when sufficiently zoomed at (a,b).





Def: For functions of 3 variables $f(x,y,z)$ at $P_0=(a,b,c)$ the lin approx is

$$\boxed{L(x,y,z) = f(a,b,c) + \underbrace{f_x(a,b,c)\,(x-a) + f_y(a,b,c)\,(y-b) + f_z(a,b,c)\,(z-c)}_{df_{(a,b,c)}}}$$

$df_{(a,b,c)}$ differential of f from Lecture XVI $\quad \begin{cases} \Delta x = x-a \\ \Delta y = y-b \\ \Delta z = z-c \end{cases}$

Example:

- Let $z = f(x,y) = 3x^2y + \sqrt{x^2+y^2}$ $\quad P_0 = (3,4)$ Find the tangent plane & equation of the linear approximation at P_0.

$f_x(x,y) = 6xy + \dfrac{x}{\sqrt{x^2+y^2}}$ $\qquad f_y(x,y) = 3x^2 + \dfrac{y}{\sqrt{x^2+y^2}}$ $\qquad f(3,4) = 113$

$f_x(3,4) = 72 + \dfrac{3}{5} = \dfrac{363}{5}$ $\qquad f_y(3,4) = 27 + \dfrac{4}{5} = \dfrac{139}{5}$.

$$\begin{cases} \text{Tangent plane:} & z = 113 + \dfrac{363}{5}(x-3) + \dfrac{139}{5}(y-3) \\ \text{Linear approx:} & L(x,y) = 113 + \dfrac{363}{5}(x-3) + \dfrac{139}{5}(y-3) \end{cases}$$

- Use linear approx. to find the value of $f(3.001, 3.99)$

$$f(3.001, 3.99) \simeq L(3.001, 3.99) = 113 + \frac{363}{5}(0.001) + \frac{139}{5}(0.01) = \boxed{113.3506}$$

§2 Differentials & change:

f differentiable at (a,b): $\Delta z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + \underbrace{\varepsilon_1(x-a) + \varepsilon_2(y-b)}_{0 \text{ as } (x,y)\to(a,b)}$

$$z = f(x,y) \rightsquigarrow \Delta z = f(x,y) - f(a,b) \approx L(x,y) - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$= f_x(a,b)\,dx + f_y(a,b)\,dy = df_{(a,b)}$$

So $\begin{cases} \Delta z : \text{rate of change in } f \text{ on the surface} \\ dz : \quad \text{" " " " } L \text{ " " tangent plane} \end{cases}$

By definition, dz is a good approximation to Δz near (a,b)

- In general: $w = f(x_1, \ldots, x_n)$, $\underline{a} = (a_1, \ldots, a_n)$, $\underline{x} = (a, \ldots, a_n)$

$dw = \dfrac{\partial f}{\partial x_1}(\underline{a})\,dx_1 + \cdots + \dfrac{\partial f}{\partial x_n}(\underline{a})\,dx_n$ is a good approximation of $\Delta w = f(\underline{x}) - f(\underline{a})$

dx_i = absolute error in measurement of quantity x_i near a_i.

$\dfrac{dx_i}{x_i}$ = relative ———————— (eg percentage)

$1\% \rightsquigarrow \dfrac{dx_i}{x_i} = 0.01$