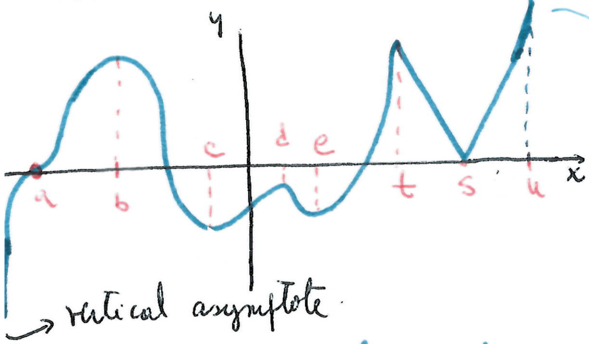
Lecture XIX (2/29/16) § 13.8 Maximum/minimum problems

Recall: Max/min problems for ONE variable functions $y = g(x)$



$y = g(x)$
- Abs. max: $x = u$
- Abs min.: none.

- Local max/min at: $x = b, c, d, e, t, s, u$ (min / max; boundary pt.)

- Critical pts: $g'(x) = 0$ or $g'(x)$ does not exist
  $g'(x) = 0 \Rightarrow x = a$; $g'(x)$ does not exist $\Rightarrow x = t, s$

- $x = a$ is a saddle point (inflexion point) (crit pt but not local max or min)
- Local max or min are critical points or end points of the interval where g is defined. ($x = u$ in the example)
- Various tests to determine when a critical pt is a local max/min:

Thm (Second derivative test for local extrema)

Suppose that f'' is continuous on an open interval containing x_0 & $f'(x_0) = 0$

(1) If $f''(x_0) > 0$ then f has a local minimum
(2) If $f''(x_0) < 0$ " f has a local maximum
(3) If $f''(x_0) = 0$, anything can happen (could be max, min or neither)

Why? (1)

	x_0^-	x_0	x_0^+
f''	>0	>0	>0
f'	$-$	0	$+$
f	decr	MIN	incr

f'' cont.

$(f')' > 0 \Rightarrow f'$ increasing around x_0

$\Rightarrow x_0$ is min

(2)

	x_0^-	x_0	x_0^+
f''	<0	<0	<0
f'	$+$	0	$-$
f	incr	MAX	decr

TODAY'S GOAL: Mimic this to study max/min of functions of 2 or more variables

§1 Local Maximum/Minimum Values

Fix $z = f(x,y) : D \longrightarrow \mathbb{R}$

Def:
- f has a local maximum value at (a,b) if $f(x,y) \leq f(a,b)$ for all (x,y) in D where (x,y) in $B_r(a,b)$ for some $r > 0$ ((x,y) in an open disk centered at (a,b))
- f has a local minimum value at (a,b) if $f(x,y) \geq f(a,b)$ for all (x,y) in D & (x,y) in $B_r(a,b)$ for some $r > 0$

If (a,b) is an interior pt of D, we know that $B_r(a,b) \subseteq D$ for some r

Note: • (a,b) is local max $\Rightarrow$ the graph of f has a peak at $(a,b,f_{(a,b)})$

 $\longleftarrow$ locally: looks like a CAP.

• (a,b) is local min $\Rightarrow$ the graph of f has a hollow at $(a,b,f_{(a,b)})$

 $\longleftarrow$ locally: looks like a CUP.

Thm: Assume f_x & f_y exist at (a,b)

If f has a local max/min at (a,b), then $\nabla f_{(a,b)} = \vec{0}$.

Proof 1: By contradiction: If $\nabla f_{(a,b)} \neq \vec{0}$ then the function increases in the direction of $\nabla f_{(a,b)}$ & decreases in the direction of $-\nabla f_{(a,b)}$. $\left(D_{\vec{u}} f_{(a,b)} > 0 \text{ if } \vec{u} = \dfrac{\nabla f_{(a,b)}}{|\nabla f_{(a,b)}|} \;\&\; D_{-\vec{u}} f_{(a,b)} < 0 \right)$
Therefore, f cannot have a local max/min at (a,b).

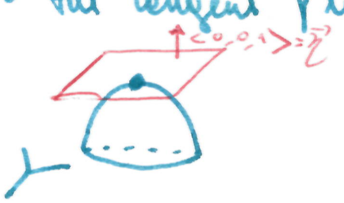
Proof 2: Write $g(x) := f_{(x,b)}$ function of 1 variable defined around $x=a$
Since f has a local max/min at (a,b), then g has a local max/min at $x=a$ &

$$g'(a) = \frac{\partial f}{\partial x}(a,b) \underset{\substack{\| \\ 0}}{} \quad (\text{local max/min of } g \text{ is a crit. pt.})$$

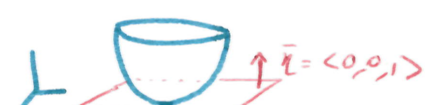
Similarly $h(y) = f_{(a,y)}$ has a local max/min at $y=b$, so $0 = h'(b) = \dfrac{\partial f}{\partial y}(a,b)$

Conclusion: $\nabla f_{(a,b)} = \vec{0}$.

• **Application**: If f is differentiable at (a,b) & (a,b) is a local max/min then $\vec{n}_{(a,b)} = \langle -f_{x\,(a,b)}, -f_{y\,(a,b)}, 1 \rangle = \langle 0, 0, 1 \rangle$ is the normal direction to the tangent plane at $(a,b,f_{(a,b)})$. So this plane is parallel to the xy-plane.



or



The plane has equation $z = \underbrace{f_{(a,b)}}_{\text{constant!}}$

Def: (a,b) is a critical point if either
(1) $\nabla f_{(a,b)} = \vec{0}$, or
(2) at least one of $f_{x\,(x,y)}$ & $f_{y\,(x,y)}$ is not defined at (a,b).

Note: • In both cases, we don't have a direction $\vec{u}$ in which $f(x,y)$ increases/decreases most rapidly.

• The Theorem says precisely that local max/min are critical pts (if they are not boundary pts of D).

Example 1 Find the critical pts of $f(x,y) = 3x^2+8y^2-2xy+4$. (differentiable!)

So: $f_x(x,y) = 6x-2y = 0$ & $f_y = 16y-2x = 0$

$\boxed{3x = y}$ & $\boxed{x = 8y}$

Only solution to BOTH equations is $x=y=0$ ⇝ Unique critical pt $=(0,0)$.

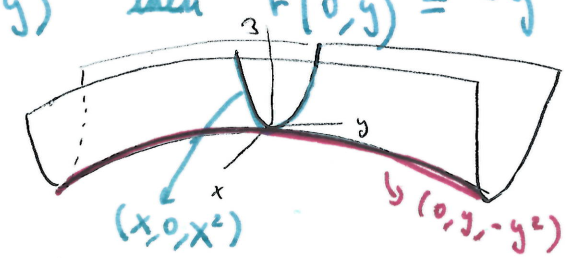
• Def: Assume (a,b) is a critical pt of f. Then (a,b) is a saddle pt if it's not a local max nor a local min. That is, in EVERY open ball centered at (a,b) there are points (x,y) for which $f(x,y) > f(a,b)$ (not local max) & points for which $f(x,y) < f(a,b)$ (not local min)

Ex 2 $z = f(x,y) = x^2-y^2$ ⇝ $\nabla f = \langle 2x, -2y\rangle$ so critical pt $=(0,0)$

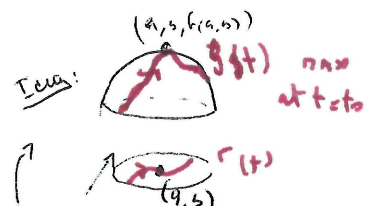
• If we take pts $(x,0)$ $(\neq (0,0))$, then $f(x,0) = x^2 > 0 = f(0,0)$

• ——— $(0,y)$ $(\neq (0,0))$ Then $f(0,y) = -y^2 < 0 = f(0,0)$

So $(0,0)$ is a saddle pt.



Idea:



§2. Second derivative test:

Observation f has a local max/min at (a,b) if and only if for every parametric curve $\vec{r}(t) = \langle x(t), y(t)\rangle$ where $\vec{r}(t_0) = \langle a,b\rangle$, the function
$$g(t) = f(\vec{r}(t)) = f(x(t), y(t))$$
has a local max/min at $t=t_0$.

Ex 3 $f(x,y) = 3x^2+8y^2-2xy+4$ from ex 1: Only crit pt $=(0,0)$

The Chain Rule gives $g'(t) = f_x(x,y)\, x'(t) + f_y(x,y)\, y'(t)$. $=0$ at $t=t_0$ ✓

Use Chain rule on f_x, f_y:
$$g''(t) = \left(f_{xx}(x,y)\, x'(t) + f_{xy}(x,y)\, y'(t)\right) x'(t) + f_x x''(t) + \left(f_{yx}(x,y)\, x'(t) + f_{yy}(x,y)\, y'(t)\right) y'(t) + f_y y''(t)$$

$f_{xy} = f_{yx}$ because they are cont ✓

(*) $$\boxed{g''(t) = f_{xx}(x,y)\cdot\left(x'(t)\right)^2 + 2 f_{xy}(x,y)\, x'(t)\cdot y'(t) + f_{yy}\left(y'(t)\right)^2 + f_x(x,y)\, x''(t) + f_y(x,y)\, y''(t)}$$

In the example: $f_{xx}(0,0) = 6$, $f_{yy}(0,0) = 16$, $f_{xy}(0,0) = -2$, $f_x = f_y = 0$

so $g''(t) = 6\, x'(t)^2 - 4\, x'(t)\, y'(t) + 16\, y'(t)^2$

$= 6\left(x'(t) - \frac{1}{3} y'(t)\right)^2 + \left(16 - \frac{6}{9}\right) y'(t)^2 > 0$ always!

So $(0,0)$ is a local minimum (because it's a local min. of $g(t)$).

• General criterion from (*): At $t = t_0$, we have $f_x(a,b) = f_y(a,b) = 0$.

$$g''(t_0) = f_{xx}(a,b)\, x'(t_0)^2 + 2 f_{xy}(a,b)\, x'(t_0)\, y'(t_0) + f_{yy}(a,b)\, y'(t_0)^2$$

Assuming $f_{xy}(a,b) = f_{yx}(a,b)$ & $f_{xx}(a,b) \neq 0$:

$$g''(t_0) = \underline{f_{xx}(a,b)}\ \underbrace{\left(x'(t_0) + \frac{f_{xy}(a,b)}{f_{xx}(a,b)}\, y'(t_0)\right)^2}_{\geq 0} + \underline{\frac{(y'(t_0))^2}{f_{xx}(a,b)}}\ \underbrace{\left(f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2\right)}_{\text{sign} = ?}$$

$f_{xx}(a,b)$ & $\frac{1}{f_{xx}(a,b)}$ have the same sign.

Theorem (Second Derivative Test) Assume all 2^{nd} order partial derivatives of f are continuous on a ball $B_r(a,b)$ inside D & $\nabla f(a,b) = \vec{0}$.

Let $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$ (discriminant of f)

(1) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b)

(2) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, ———— local minimum at (a,b)

(3) If $D(a,b) < 0$, then f has a saddle pt at (a,b)

(4) If $D(a,b) = 0$, anything can happen.

Note: D is the determinant of the Hessian matrix $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ where $f_{xy} = f_{yx}$!

Examples Recitation 07.

Remark: We can see local max/min & saddle points from the contour level (drawing of all level curves together)