Lecture XX (03/02/16) §13.8 Absolute max/min, §13.9 Lagrange Multipliers

Last Time: Studied how to find local max/min & saddle points of $f: D \longrightarrow \mathbb{R}$, $f = f(x,y)$

Key: Critical points of f : either (1) $\nabla f(a,b) = \vec{0}$
(a,b) in D.
(2) One of f_x & f_y is not defined at (a,b)

• Second Derivative Test : help decide if a crit pt is local max/min or neither

Q: What about abs. max/min, ie extreme values of f over D?

§13.8. Absolute maximum & minimum values

Def: (1) Fix (a,b) in D. If $f(x,y) \geq f(a,b)$ for EVERY (x,y) in D, then $f(a,b)$ is an absolute minimum value of f in D.

(2) If $f(x,y) \leq f(a,b)$ for EVERY (x,y) in D, then $f(a,b)$ is an absolute maximum value of f in D.

Thm: If f is continuous and D is closed and bounded (meaning, we can find $R>0$ where $D \subseteq B_R(0,0)$, i.e. D lies in the ball centered at $(0,0)$ of radius R), then f has absolute max & min values.

(Analogous to the case of one variable functions: $f: [c,d] \longrightarrow \mathbb{R}$ continuous, then f attained max & min values).

Question: How to find them? (Closed = D contains all its boundary points)

Proposition: $f: D \longrightarrow \mathbb{R}$ continuous, D in $\mathbb{R}^2$ closed and bounded. Then we determine the absolute max/min values of f on D as follows:

① Find all critical points of f in D (~ candidates for local max/min)

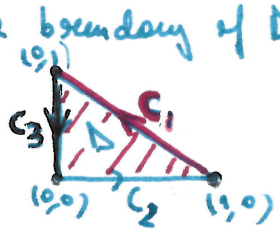
② Find the max/min values of f on the boundary of D

③ Compare the values obtained in ① & ② : the largest (resp. lowest) gives the absolute max (resp. min.) value.

• Techniques for ② :

(1) The boundary of D is a union of parametric curves. If we restrict f to each curve, we have a max/min problem for functions of 1 var.

Eg



Eg: • C_1 is $\vec{r}_1(t): [0,1] \longrightarrow \mathbb{R}^2$ $\quad \vec{r}_1(t) = t\langle 0,1\rangle + (1-t)\langle 1,0\rangle$

• f on C_1: $g(t) = f(\vec{r}_1(t)) = f(1-t, t): [0,1] \longrightarrow \mathbb{R}$

$f(x,y) = xy - 49x - 47y^2 + 13$ ...

$\Rightarrow g(t) = (1-t)t - 49(1-t) - 47t^2 + 13 \; = \; -48t^2 + 48t - 36$ for $0 \le t \le 1$

• Check critical values of g: $g'(t) = -96t + 48 = 0 \rightsquigarrow t = \frac{1}{2}$ in $[0,1]$ only crit value
& $g(\frac{1}{2}) = \boxed{-24}$

• Evaluate at end points & compare:

$g(0) = \boxed{-36}$, $g(1) = -48 + 48 - 36 = \boxed{-36}$
$\overset{\shortparallel}{f(1,0)}$ $\quad$ $\overset{\shortparallel}{f(0,1)}$

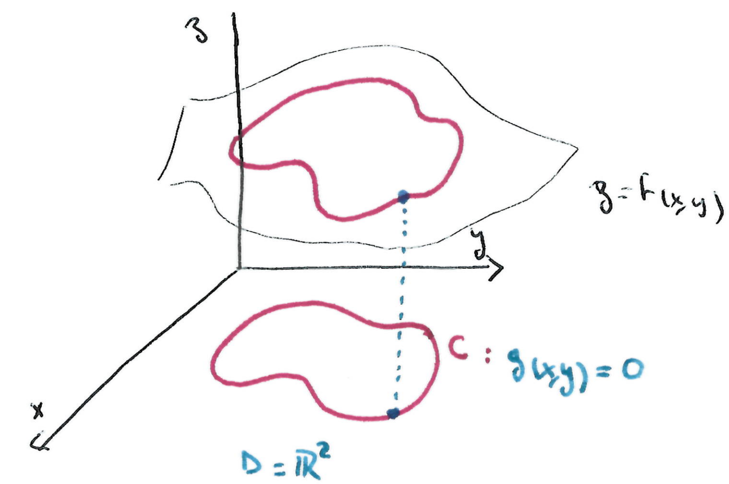
$\rightsquigarrow \begin{cases} \text{max on } \vec{r}(\frac{1}{2}) = \langle \frac{1}{2}, \frac{1}{2} \rangle \\ \text{min on } \vec{r}(0) = \langle 1, 0 \rangle. \\ \;\;\text{"} \quad \text{"} \quad \vec{r}(1) = \langle 0, 1 \rangle \end{cases}$ over C_1

We find the max / min on each curve & compare the values to find the winner on the boundary.

(2) What if we cannot find a parametrization of each curve but instead we have an equation? $\rightsquigarrow$ Lagrange multipliers!

Remark: If D is open (eg $B_3(0,0) = \{(x,y) : x^2+y^2 < 9\}$) or unbounded (eg $\{(x,y) \mid x \ge 0\}$), there is no general procedure because absolute max or min values may not exist. (Examples: Recitation 07)

§13.9. Lagrange multipliers:



GOAL: Maximize / minimize a differentiable function $f : D \longrightarrow \mathbb{R}$ $\quad f(x_1, \dots, x_n)$

SUBJECT TO constraints:

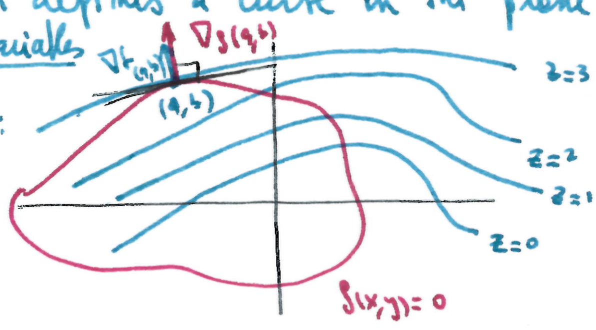
$\begin{cases} g_1(x_1, \dots, x_n) = 0 \\ \vdots \\ g_s(x_1, \dots, x_n) = 0 \end{cases}$

where $g_1, \dots, g_s$ are differentiable.

TODAY: $\begin{cases} n=2 \\ n=3 \end{cases}$, $s = 1$ $\quad$ Max/minimize $f = f(x,y) : D \longrightarrow \mathbb{R}$ subject to $g(x,y) = 0$, which defines a curve in the plane.

§1 Two variables

Idea:



• Assume (a,b) in D is max of f over $g(x,y) = 0$

$\begin{cases} z = f(x,y) : \text{Draw the level curves of } f. \\ g(x,y) = 0 \text{ is also a level curve} \end{cases}$

• $\nabla g(a,b) \perp$ Tangent dir to the curve $g(x,y) = 0$ at (a,b)

• The curve touches the level curve $(z=3)$ but doesn't cross it.

So the two level curves $f(x,y)=3$ & $g(x,y)=0$ have the same tangent line at (a,b).

- $\nabla f_{(a,b)} \perp$ tangent dir to the curve $(z=3)$ at (a,b)

Conclusion: $\nabla f_{(a,b)}$ & $\nabla g(a,b)$ must be proportional!.

Theorem (Lagrange multipliers in 2 variables)

Fix $\underset{(x,y)}{f}: D \longrightarrow \mathbb{R}$ differentiable and D contains the curve C given by $g(x,y)=0$, where g is differentiable. Assume (a,b) is a local extreme value of f on C. Then

- (1) $\nabla f_{(a,b)}$ is perpendicular to the tangent line to C at (a,b).
- (2) If $\nabla g(a,b) \neq \vec{0}$, then there is λ in $\mathbb{R}$ (Lagrange multiplier) such that

$$(*) \quad \boxed{\nabla f_{(a,b)} = \lambda \nabla g_{(a,b)}}$$

Proof: Locally around (a,b), we can write C as a parametric curve $\vec{r}_{(t)} = \langle x_{(t)}, y_{(t)} \rangle$ with $\vec{r}(t_0) = \langle a,b \rangle$. and $\vec{r}'(t_0) = \langle x'(t_0), y'(t_0) \rangle$

Using the chain rule on $h(t) = f(x(t), y(t))$ we get

$$h'(t_0) = f_x(a,b)\, x'(t_0) + f_y(a,b)\, y'(t_0) = \nabla f_{(a,b)} \cdot \vec{r}'(t_0).$$

By definition $h(t)$ has a local extrema at $t=t_0$ & h is differentiable at $t=t_0$, so $0 = h'(t_0) = \nabla f_{(a,b)} \cdot \vec{r}'(t_0)$. The claim (1) follows.

- For (2): $\nabla g(a,b)$ is also orthogonal to C (because C is a level curve of g containing (a,b)). Since $\nabla g(a,b) \neq \vec{0}$, then $\nabla g(a,b)$ & $\nabla f_{(a,b)}$ are parallel & we can find λ as in the statement

(Notice that λ exists even if $\nabla f_{(a,b)} = \vec{0}$).

- Meaning of The Thm? We can find λ and (a,b) satisfying $\begin{cases} f_x(a,b) = \lambda g_x(a,b) \\ f_y(a,b) = \lambda g_y(a,b) \\ g(a,b) = 0 \end{cases}$

These will give candidates for abs max/min which

- Then, compare to pick the winners (they exist if C is a bounded and closed curve)

Example: Find the max & min values of $f_{(x,y)} = x^2y^2$ subject to $2x^2+y^2=1$

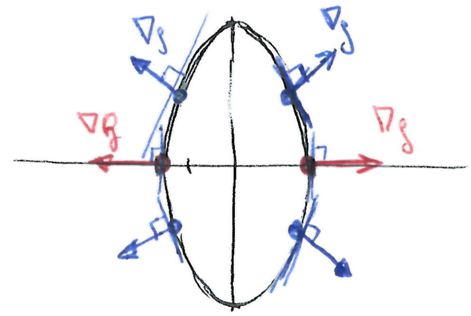
Soln: Write $g(x,y) = 2x^2+y^2-1$ ⇝ C is an ellipse, so closed & bounded

So the function f has a max/min on C.

$$\nabla f(x,y) = \langle 2xy^2, 2yx^2\rangle = 2xy\langle y,x\rangle \; ; \; \nabla g(x,y) = \langle 4x, 4y\rangle = 4\langle x,y\rangle$$

We must solve 3 eqns at the same time:

$$\begin{cases} 2xy^2 = 4\lambda x & (1) \\ 2yx^2 = 4\lambda y & (2) \\ 2x^2+y^2 = 1 & (3) \end{cases}$$



From (3): $y^2 = 1-2x^2$ & plug in (1)

$$2x(1-2x^2) = 4\lambda x \iff 2x(2\lambda - 1 + 2x^2) = 0$$

• If $x=0 \Rightarrow y = \pm 1$ (from (3)) & $4\lambda y = 0$ (from (2)) $\Rightarrow \lambda = 0$.

Check $\nabla g(0,\pm 1) = \langle 0, \pm 4\rangle \neq \vec{0}$, $\nabla f(0,\pm 1) = \vec{0}$ so $\lambda = 0$ works.

• Assume $x \neq 0$, then $2\lambda = 1-2x^2$ so $\lambda = \frac{1-2x^2}{2}$

We replace in (2): $2yx^2 \overset{?}{=} 4\left(\frac{1-2x^2}{2}\right)y = 2y(1-2x^2)$ (*)

• If $y=0$, then $x = \pm\frac{1}{\sqrt{2}}$ (from (3)) & $\lambda = 0$; $\nabla f(\pm\frac{1}{\sqrt{2}}, 0) = \vec{0}$ ✓

$\nabla g(\pm\frac{1}{\sqrt{2}}, 0) = \langle \pm\frac{4}{\sqrt{2}}, 0\rangle = \vec{0}$

so $y \neq 0$, from (*) we get $x^2 = 1-2x^2$

$x^2 = \frac{1}{3} \to x = \pm\frac{1}{\sqrt{3}}$ & then $y = \pm\frac{1}{\sqrt{3}}$

$\boxed{\lambda = \frac{1}{6}}$ (4 candidates!)

$$\begin{cases} \nabla f(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \langle \frac{\pm 2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\rangle \\ \nabla g(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \langle \frac{\pm 4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\rangle \neq \vec{0} \end{cases}$$ & $\lambda = \frac{1}{6}$ works for both

$$\begin{cases} \nabla f(\pm\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}) = \langle \frac{\pm 2}{3\sqrt{3}}, \frac{\mp 2}{3\sqrt{3}}\rangle \\ \nabla g(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \langle \frac{\pm 4}{\sqrt{3}}, \frac{-4}{\sqrt{3}}\rangle \neq \vec{0} \end{cases}$$ & $\lambda = 1/6$ works for both :

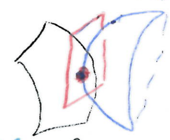
We get 5 candidate solutions: We evaluate f at these 5 pts to decide the winners:

$f(\pm\frac{1}{\sqrt{2}}, 0) = 0$ MIN ; $f(\pm\frac{1}{\sqrt{3}}, +\frac{1}{\sqrt{3}}) = f(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}) = \frac{1}{9}$ MAX

§2 Three variables:

Here, the role of tangent lines is replaced by tangent planes & level curves are replaced by level surfaces.

Key: The level surface $g(x,y,z)=0$ & the level surface $f(x,y,z)=M$ (where $M=f(a,b,c)$ is a local extreme value & $g(a,b,c)=0$) share the same tangent plane at (a,b,c).



- $\nabla g(a,b,c)$ is the normal direction to the tangent plane at (a,b,c)
- $\nabla f(a,b,c)$ —"—"—"—"— at (a,b,c).

$\Rightarrow$ These 2 gradients must be proportional if $\nabla g(a,b,c)\neq\vec{0}$.

Theorem (Lagrange multipliers in 3 variables)

Fix $f=f(x,y,z): D\longrightarrow\mathbb{R}$ differentiable and D contains the surface S given by $g(x,y,z)=0$ where g is differentiable. Assume (a,b,c) is a local extreme value of f on S. Then:

(1) $\nabla f(a,b,c)$ is the normal direction of the tangent plane to S at (a,b,c)

(2) If $\nabla g(a,b,c)\neq\vec{0}$, then there is λ in $\mathbb{R}$ (Lagrange multiplier) such that

$$\boxed{\nabla f(a,b,c)=\lambda\nabla g(a,b,c)}$$

- Strategy to find absolute max / min values when S is closed & bounded:

① Find the local max/min: (a,b,c) & λ satisfy 4 eqns below:

$$\left\{\begin{array}{l} f_x(a,b,c)=\lambda g_x(a,b,c)\\ f_y(a,b,c)=\lambda g_y(a,b,c)\\ f_z(a,b,c)=\lambda g_z(a,b,c)\\ g(a,b,c)=0\end{array}\right.\quad \left.\begin{array}{l}\\ \\ \\ \end{array}\right\}\equiv \nabla f(a,b,c)=\lambda\nabla g(a,b,c)$$

- Find (a,b,c) where ($g(a,b,c)=0$ & $\nabla g(a,b,c)=\vec{0}$)

② Compare the values of f at all points found in ② & pick the winners.