Last time: Studied how to find local max/min and saddle points of \( f: D \rightarrow \mathbb{R} \) if \( f(x, y) \) is define in \( D \).

Key: Critical points of \( f \) : either (1) \( \nabla f(x, y) = \mathbf{0} \)
(2) \( f_x \) or \( f_y \) is not defined at \( (x, y) \).

Second derivative Test : help decide if a crit pt is local max/min or neither.

Q: What about abs. max/min, ie extrema values of \( f \) over \( D \)?

\( \S 13.8 \). Absolute maximum & minimum values

\[ Q1 \]: Fix \( (a, b) \) in \( D \). If \( f(x, y) \geq f(a, b) \) for every \( (x, y) \) in \( D \), then \( f(a, b) \) is an absolute minimum value of \( f \) in \( D \).

\[ Q2 \]: If \( f(x, y) \leq f(a, b) \) for every \( (x, y) \) in \( D \), then \( f(a, b) \) is an absolute maximum value of \( f \) in \( D \).

Thus: \( f \) is continuous and \( D \) is closed and bounded (meaning, we can find \( R > 0 \) when \( D \subseteq B_r(0, 0) \), i.e., \( D \) lies in the ball centered at \((0, 0)\) of radius \( R \)), then \( f \) has absolute max & min values.

(Analogous to the case of one variable function: \( f: [c, d] \rightarrow \mathbb{R} \) continuous, then \( f \) attained max & min values).

Question: How to find them? (Closed = \( D \) contains all its boundary points)

Proposition: \( f: D \rightarrow \mathbb{R} \) continuous, \( D \) in \( \mathbb{R}^2 \) closed and bounded. Then we determine the absolute max/min values of \( f \) on \( D \) as follows:

1. Find all critical points of \( f \) in \( D \) (no candidates for local max/min).
2. Find the max/min values of \( f \) on the boundary of \( D \).
3. Compare the values obtained in (1) & (2): the largest (resp. lowest) gives the absolute max (resp. min) value.

Techniques \( \S 17 \): 2

\( 1 \): The boundary of \( D \) is a union of parametric curves. If we restrict \( f \) to each curve, we have a max/min problem for functions of 1 var.

\[ \text{Eg: } C_1 \text{ is } f_1(t): [0, 1] \rightarrow \mathbb{R}^2 \quad f_1(t) = t \begin{pmatrix} 0 \ 1 \end{pmatrix} + (1-t) \begin{pmatrix} 1 \ 0 \end{pmatrix} \]

\[ \text{Eg: } f \circ f_1(t): [0, 1] \rightarrow \mathbb{R} \]

\[ \text{Eg: } f \circ f_1(t): [0, 1] \rightarrow \mathbb{R} \]
\[ f(x,y) = x^2 - 8x + 2y^2 + 13 \]

\[ \Rightarrow g(t) = (1-t) - 98 (1-t)^2 + 13 = -98 t^2 + 98 t - 26 \quad \text{for } 0 \leq t \leq 1 \]

- Check critical values of \( g \):
  \[ g'(t) = -196 t + 98 = 0 \Rightarrow t = \frac{1}{2} \quad \text{in } [0,1] \]
  \[ g\left(\frac{1}{2}\right) = -24 \]

Evaluate at end points & compare:

- \( g(0) = -26 \), \( g(1) = -98 + 98 - 36 = -36 \)

\[ \max \text{ occurs at } t = \frac{1}{2} \quad \text{on } C, \]
\[ \min \text{ occurs at } t = 1 \]

We find the max/min on each curve & compare the values to find the winner on the boundary.

(2) What if we cannot find a parametrization of each curve but instead we have an equation? Use Lagrange multipliers!

**Remark:** If \( D \) is open (e.g., \( B_3 (0,0) = \{(x,y): x^2 + y^2 < 9\} \)), no bounded (e.g. \( B(0,0) : x \geq 0 \}), there is no general procedure because absolute max/min values may not exist. (Example: Recitation 07)

§ 13.9. **Lagrange multipliers:**

**Goal:** Maximize/Minimize a function

\[ f: D \rightarrow \mathbb{R} \quad f(x_1, \ldots, x_n) \]

Subject to constraints:

\[ \begin{cases} g_1(x_1, \ldots, x_n) = 0 \\ \vdots \\ g_s(x_1, \ldots, x_n) = 0 \end{cases} \]

where \( g_1, \ldots, g_s \) are differentiable.

Today: \( m = 2 \), \( s = 1 \)

Max/minimize which defines a curve in the plane.

**Idea:**

- **ASSUMED:** \( D \) is open and \( f \) is differentiable on \( D \)
- \( g(x,y) = 0 \)
- The curve \( \{ (x,y): g(x,y) = 0 \} \)
- \( \nabla g(a,b) \perp \) Tangent dir to the curve \( g(x,y) = 0 \)
- \( \nabla f(a,b) \) points in the direction of the steepest increase of \( f \) at \( (a,b) \)
- \( f \) is constant along level curves of \( f \)
- \( f \) is constant along level curves of \( g \)
- \( \nabla f(a,b) \cdot \nabla g(a,b) = 0 \)
so the two level curves \( f(x,y) = 3 \) and \( g(x,y) = 0 \) have the same tangent line at \((9,5)\).

- \( \nabla f(9,5) \perp \) tangent line to the curve \((z = 3)\) at \((9,5)\).

\[ \nabla f(9,5) \perp \nabla g(9,5) \] must be proportional!

**Theorem (Lagrange multipliers in 2 variables)**

Fix \( f : D \rightarrow \mathbb{R} \) differentiable and \( D \) contains the curve \( g(x,y) = 0 \), where \( g \) is differentiable. Assume \((9,5)\) is a local extreme value of \( f \) on \( C \). Then:

1. \( \nabla f(9,5) \) is perpendicular to the tangent line to \( C \) at \((9,5)\).
2. If \( \nabla g(9,5) \neq \overrightarrow{0} \), then there is \( \lambda \) in \( \mathbb{R} \) (Lagrange multipliers) such that

\[ \nabla f(9,5) = \lambda \nabla g(9,5) \]

**Proof:** Locally around \((9,5)\), we can write \( C \) as a parametric curve \( \overrightarrow{r}(t) = \langle x(t), y(t) \rangle \) with \( \overrightarrow{r}(t_0) = \langle 9, 5 \rangle \) and \( \overrightarrow{r}'(t_0) = \langle x'(t_0), y'(t_0) \rangle \).

Using the chain rule on \( f(\overrightarrow{r}(t)) = f(x(t), y(t)) \), we get

\[ h'(t) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t) = \nabla f(9,5) \cdot \overrightarrow{r}'(t_0). \]

By definition, \( h(t) \) has a local extrema at \( t = t_0 \) and \( h \) is differentiable at \( t = t_0 \) so

\[ 0 = h'(t_0) = \nabla f(9,5) \cdot \overrightarrow{r}'(t_0). \] The claim (1) follows.

For (2): \( \nabla g(9,5) \) is also orthogonal to \( C \) (because \( C \) is a level curve of \( g \)). Since \( \nabla g(9,5) \neq \overrightarrow{0} \), then \( \nabla g(9,5) \perp \nabla f(9,5) \) are parallel and we can find \( \lambda \) as in the statement (Notice that \( \lambda \) exists even if \( \nabla f(9,5) = \overrightarrow{0} \)).

**Meaning of the Theorem:** We can find \( \lambda \) satisfying

\[
\begin{align*}
\left\{ \begin{array}{l}
x(x,y) = \lambda y (x,y) \\
y(x,y) = \lambda x (y, x) \\
g(x,y) = 0
\end{array} \right.
\]

There will give candidates for the max/min which

Then, compare to pick the winners (they exist if \( C \) is a bounded and closed curve).

**Example:** Find the max and min values of \( f(x,y) = x^2 y^2 \)

subject to \( 2x^2 + y^2 = 1 \).
**Soln:** Write \( g(x,y) = 2x^2 + y^2 - 1 \Rightarrow C \) is an ellipse, so closed & bounded

So the function \( f \) has a max/min on \( C \).

\[
\nabla f(x,y) = < 2x, 2y > = 2xy <\frac{x}{y}, \frac{y}{x}>; \quad \nabla g(x,y) = < 4x, 4y > = 4 <x, y>
\]

We must solve 3 eqns at the same time:

\[
\begin{cases}
  2xy^2 = 4\lambda x \\
  2y^2 = 4\lambda y \\
  2x^2 + y^2 = 1
\end{cases}
\]

From (3): \( y^2 = 1 - 2x^2 \) & plug in (1):

\[
2x(1 - 2x^2) = 4\lambda x \Rightarrow 2x(2\lambda - 1 + 2x^2) = 0
\]

- If \( x = 0 \Rightarrow y = \pm 1 \) (from (3)) & \( 4\lambda y = 0 \) (from (2)) \( \Rightarrow \lambda = 0 \).

  \[ \nabla g(0, \pm 1) = < 0, \pm 4 > \not= \vec{0} \quad \nabla f(0, \pm 1) = \vec{0} \quad \text{so} \quad \lambda = 0 \text{ works.} \]

- Assume \( x \neq 0 \), then \( 2\lambda = 1 - 2x^2 \Rightarrow \lambda = \frac{1 - 2x^2}{2} \)

  We replace in (2):

  \[
  2y^2 = 4 \left( \frac{1 - 2x^2}{2} \right) y = 2y(1 - 2x^2) \]

  - If \( y = 0 \), then \( x = \pm \sqrt{\frac{1}{2}} \) (from (3)) & \( \lambda = 0 \):

    \[
    \nabla f\left(\pm \frac{\sqrt{2}}{\sqrt{2}}, 0\right) = \vec{0} \quad \nabla g\left(\pm \frac{\sqrt{2}}{\sqrt{2}}, 0\right) = < \pm 4, 0 > = \vec{0}
    \]

  - So \( y \neq 0 \), from (x) we get \( x^2 = 1 - 2x^2 \)

    \[
    x^2 = \frac{1}{3} \Rightarrow x = \pm \frac{1}{\sqrt{3}} \quad \text{& then} \quad y = \pm \frac{1}{\sqrt{3}}
    \]

    \[
    \lambda = \frac{1}{6}
    \]

    \[
    \begin{cases}
    \nabla f\left(\pm \frac{\sqrt{1}}{\sqrt{3}}, \pm \frac{\sqrt{3}}{\sqrt{3}}\right) = < \pm \frac{\sqrt{2}}{3}, \pm \frac{2}{3} > \\
    \nabla g\left(\pm \frac{\sqrt{1}}{\sqrt{3}}, \pm \frac{\sqrt{3}}{\sqrt{3}}\right) = < \pm \frac{4}{\sqrt{3}}, \pm \frac{4}{\sqrt{3}} > \not= \vec{0}
    \end{cases}
    \]

We get 5 solutions: We evaluate \( f \) at these 5 pts. to decide the winners.

- For candidate

  \[
  f\left(\frac{\sqrt{1}}{\sqrt{3}}, 0\right) = 0 \quad \text{min} \quad f\left(\pm \frac{\sqrt{1}}{\sqrt{3}}, \pm \frac{\sqrt{3}}{\sqrt{3}}\right) = f\left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right) = \frac{1}{9} \quad \text{max}
  \]
§2 Three variables:

Here, the role of tangent lines is replaced by tangent planes. Level curves are replaced by level surfaces.

Key: The level surface \( g(x, y, z) = 0 \) & the level surface \( f(x, y, z) = N \) (where \( N = f(a, b, c) \) is a local extreme value & \( g(a, b, c) = 0 \)) share the same tangent plane at \((a, b, c)\).

\[ \nabla g(a, b, c) \] is the normal direction to the tangent plane of \( f(a, b, c) \).

\[ \nabla f(a, b, c) \]

\( \Rightarrow \) These 2 gradients must be proportional if \( \nabla g(a, b, c) \neq \mathbf{0} \).

**Theorem (Lagrange multipliers in 3 variables)**

Fix \( f(x, y, z) : D \to \mathbb{R} \) differentiable and \( D \) contains the surface \( S \) given by \( g(x, y, z) = 0 \) where \( g \) is differentiable. Assume \((a, b, c)\) is a local extreme value of \( f \) on \( S \). Then:

1. \( \nabla f(a, b, c) \) is the normal direction of the tangent plane to \( S \) at \((a, b, c)\)
2. If \( \nabla g(a, b, c) \neq \mathbf{0} \), then there is \( \lambda \) in \( \mathbb{R} \) (Lagrange multiplier) such that

\[ \nabla f(a, b, c) = \lambda \nabla g(a, b, c) \]

**Strategy to find absolute max/min values when S is closed & bounded:**

1. Find the local max/min: \((a, b, c)\) & \( \lambda \) satisfy 4 equations:

\[
\begin{align*}
    f_x(a, b, c) &= \lambda g_x(a, b, c) \\
    f_y(a, b, c) &= \lambda g_y(a, b, c) \\
    f_z(a, b, c) &= \lambda g_z(a, b, c) \\
    g(a, b, c) &= 0
\end{align*}
\]

2. Find \((a, b, c)\) where \( g(a, b, c) = 0 \) & \( \nabla g(a, b, c) = \mathbf{0} \)

3. Compare the values of \( f \) at all points found in 2 & pick the winners.